# Optimal growth, bequests and competitive equilibrium cycles in two-sector OLG models* 

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This paper is dedicated to Pierre Cartigny (1946-2019)

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#### Abstract

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[^0]
## 1 Introduction

It has been recently proved by Piketty [14] that in a country like France the annual flow of inheritance was about $20-25 \%$ of national income between 1820 and 1910, down to less than $5 \%$ in 1950 , and back up to about $15 \%$ by 2010. The following graph indeed shows a long-run cyclic behavior of inheritance flows.


Figure 1: Annual inheritance flow as a fraction of national income, France 1820-2008 (Source: Piketty [14])

Similar conclusions have been reached by Atkinson [1] for the UK as shown in the following graph:


Figure 2: Comparison of France (red) and the United Kingdom (blue): transmitted wealth as percentage of net national income from 1896 to 2008 (Source: Atkinson [1])

The objective of this paper is to provide a simple model that can explain such a long-run cyclic behavior. The standard model that allows to study inheritance flows across generations has been initially provided by Barro [3] with the concept of optimal bequest. As shown by Weil [18], as long as bequests are strictly positive across generations, the solution of the Barro model is equivalent to the solution of a Ramsey-type optimal growth model where a central planner maximizes the total intertemporal welfare.

Building on the well-known stability properties of the aggregate Ramsey model, it can be easily shown that if the life-cycle utility function of a representative generation living over two periods is additively separable, then the optimal path monotonically converges toward the steady state. In such a case there is no room for any cyclic behavior of bequests. But Michel and Venditti [13] have proved that if the life-cycle utility function is non-additively separable with a positive cross derivative across periods then endogenous period-two cycles can occur. This conclusion shows that such a model based on a preference mechanism is formally equivalent to a standard two-sector optimal growth model where period-two endogenous cycles rely on a technology mechanism as they occur if the consumption good is more capital intensive than the investment good (see Benhabib and Nishimura [6]). The main critic of this result with respect to our goal to describe accurately the long run dynamics of bequests is that period-two cycles implies negative auto-correlations of variables which are not in line with the empirical properties of macroeconomic time series.

The strategy in this paper is then to extend the Michel and Venditti [13] formulation to a two-sector economy. Beside introducing in the analysis both mechanisms relying on preference and technology, the extended model leads now to a dimension-four dynamical system which can give rise to the existence of quasi-periodic optimal paths, through the occurrence of complex characteristic roots, that do not imply negative auto-correlation of variables and are in line with the long run empirical properties of aggregate time series. The analysis is divided in two parts. In a first part, under the assumption of a non-strictly concave utility function, we show that the preference and technology mechanisms can be separated and lead, each of them, to the existence of period-two cycles. The global dynamics can then be described as the product of two cycles implying complex properties of the optimal path. In a second part, considering a strictly concave utility function, the preference and technology mechanisms are now combined and can lead to the existence of quasi-periodic cycles if the life-cycle utility function is nonadditively separable with a positive cross derivative across periods and the consumption good is more capital intensive than the investment good. We
also show that all these results are of course compatible with the conditions of positive bequests.

The paper is organized as follows. In Section 2 we present the twosector model with non-additively separable preferences, define the optimal growth problem of the central planner, prove the existence of a steady state and derive the characteristic polynomial from which the stability analysis if conducted. The existence of period-two cycles under the assumption of a non-strictly concave utility function is discussed in Section 3 together with the presentation of a simple example to illustrate the main conditions. Section 4 contains the extension to the case of a strictly concave utility function. We provide general sufficient conditions that rule out the existence of complex characteristic roots and we consider a specific utility function formulation to prove the possible existence of a Hopf bifurcation and thus of quasi-periodic cycles. In Section 5 we show that all our previous conditions are compatible with strictly positive bequests. Concluding comments are provided in Section 6 and all the proofs are contained into a final Appendix.

## 2 The model

### 2.1 Production

We consider a two-sector economy with one pure consumption good $y_{0}$ and one capital good $y$. Each good is produced with a standard constant returns to scale technology:

$$
y_{0}=f^{0}\left(k_{0}, l_{0}\right), \quad y=f^{1}\left(k_{1}, l_{1}\right)
$$

with $k_{0}+k_{1} \leq k, k$ being the total stock of capital, and $l_{0}+l_{1} \leq 1$, the total amount of labor being normalized to 1 .
Assumption 1. Each production function $f^{i}: \mathbb{R}_{+}^{2} \rightarrow \mathbb{R}_{+}, i=0,1$, is $C^{2}$, increasing in each argument, concave, homogeneous of degree one and such that for any $x>0, f_{k_{i}}^{i}(0, x)=f_{l_{i}}^{i}(x, 0)=+\infty, f_{k_{i}}^{i}(+\infty, x)=f_{l_{i}}^{i}(x,+\infty)=$ 0 .

For any given $(k, y, \ell)$, we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$
\begin{align*}
T(k, y)=\max _{k_{0}, k_{1}, l_{0}, l_{1}} & f^{0}\left(k_{0}, l_{0}\right) \\
\text { s.t. } & y \leq f^{1}\left(k_{1}, l_{1}\right) \\
& k_{0}+k_{1} \leq k  \tag{1}\\
& l_{0}+l_{1} \leq 1 \\
& k_{0}, k_{1}, l_{0}, l_{1} \geq 0
\end{align*}
$$

The value function $T(k, y)$ is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that $T(k, y)$ is concave non strictly. We will assume in the following that $T(k, y)$ is at least $C^{2}$.

Denoting $p$ the price of the investment good, $r$ the rental rate of capital and $w$ the wage rate, all in terms of the price of the consumption good, it is easy to show that

$$
\begin{equation*}
T_{k}(k, y,)=r(k, y), \quad T_{y}(k, y)=-p(k, y) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
w(k, y)=T(k, y)-r(k, y) k+p(k, y) y \tag{3}
\end{equation*}
$$

We can also characterize the second derivatives of $T(k, y)$. From the concavity property we have:

$$
T_{k k}(k, y)=\frac{\partial r}{\partial k} \leq 0, T_{y y}(k, y)=-\frac{\partial p}{\partial y} \leq 0
$$

As shown by Benhabib and Nishimura [7], the sign of the cross derivative $T_{k y}(k, y)$ is given by the sign of the relative capital intensity difference between the two sectors. Denoting

$$
a_{00}=l_{0} / y_{0}, \quad a_{10}=k_{0} / y_{0}, \quad a_{01}=l_{1} / y, \quad a_{11}=k_{1} / y
$$

the capital and labor coefficients in each sector, it is easy to derive from the constant returns to scale property that

$$
\begin{equation*}
\frac{d p}{d r}=a_{01}\left(\frac{a_{11}}{a_{01}}-\frac{a_{10}}{a_{00}}\right) \equiv b \tag{4}
\end{equation*}
$$

with $b$ the relative capital intensity difference, and thus

$$
T_{k y}=T_{y k}=-\frac{\partial p}{\partial r} \frac{\partial r}{\partial k}=-T_{k k} b
$$

The sign of $b$ and of $T_{k y}$ is positive if and only if the investment good is capital intensive. Notice also that $T_{y y}(k, y)$ may be written as

$$
T_{y y}=-\frac{\partial p}{\partial r} \frac{\partial r}{\partial y}=T_{k k} b^{2}
$$

Remark: The derivative $d r / d p=b^{-1}$ is well-known in trade theory as the Stolper-Samuelson effect. Similarly, at constant prices, we can derive the associated Rybczinsky effect

$$
\frac{d y}{d k}=b^{-1}
$$

We therefore find the well-known duality between the Rybczinsky and Stolper-Samuelson effects.

### 2.2 Preferences

The economy is populated by a constant population of finitely-lived agents. ${ }^{1}$ In each period $t, N_{t}=N$ persons are born, and they live for two periods: they work during the first (with one unit of labor supplied) and they have preferences for consumption ( $c_{t}$, when they are young, and $d_{t+1}$, when they are old) which are summarized by the utility function $u\left(c_{t}, B d_{t+1}\right)$, with $B>0$ a normalization constant, such that

Assumption 2. $u(c, B d)$ is increasing with respect to each argument $\left(u_{1}(c, B d)>0\right.$ and $\left.u_{d}(c, B d)>0\right)$, concave and $C^{2}$ over the interior of $\mathbb{R}_{+}^{2}$. Moreover, for all consumption levels $c, d>0, u_{c}(0, B d)=u_{d}(c, 0)=\infty$ and $u_{c}(+\infty, B d)=u_{d}(c,+\infty)=\infty$.

We also introduce a standard normality assumption between the two consumption levels

Assumption 3. Consumptions c and d are normal goods.
We finally introduce the following useful elasticities of substitution of consumptions:

$$
\begin{align*}
& \epsilon_{c c}=-u_{c} / u_{c c} c>0, \quad \epsilon_{c d}=-u_{c} / u_{c d} B d,  \tag{5}\\
& \quad \epsilon_{d c}=-u_{d} / u_{c d} c, \quad \epsilon_{d d}=-u_{d} / u_{d d} B d>0 \tag{6}
\end{align*}
$$

Notice that the normality Assumption 3 implies $1 / \epsilon_{c c}-1 / \epsilon_{d c} \geq 0$ and $1 / \epsilon_{d d}-$ $1 / \epsilon_{c d} \geq 0$ and concavity in Assumption 2 implies $1 /\left(\epsilon_{c c} \epsilon_{d d}\right)-1 /\left(\epsilon_{d c} \epsilon_{c d}\right) \geq 0$.

### 2.3 The optimal growth problem

Under complete depreciation within one period, ${ }^{2}$ the capital accumulation equation is

$$
\begin{equation*}
k_{t+1}=y_{t} \tag{7}
\end{equation*}
$$

Total labor being normalized to 1 , we consider from now on that $N=1$. At each time $t$ total consumption is then given by the social production function, i.e. $c_{t}+d_{t}=T\left(k_{t}, y_{t}\right)$. The objective of the central planner combines utilities of successive generations

$$
\begin{equation*}
\max _{\left\{c_{t}, d_{t+1}\right\}} \sum_{t=0}^{+\infty} \beta^{t} u\left(c_{t}, B d_{t+1}\right) \tag{8}
\end{equation*}
$$

[^1]where $\beta \in(0,1]$ is the discount factor. ${ }^{3}$ Considering (7) and the fact that $c_{t}=T\left(k_{t}, y_{t}\right)-d_{t}$, the optimization program (8) can be equivalently written as follows
\[

$$
\begin{equation*}
\max _{\left\{d_{t+1}, k_{t+1}\right\}} \sum_{t=0}^{+\infty} \beta^{t} u\left(T\left(k_{t}, k_{t+1}\right)-d_{t}, B d_{t+1}\right) \tag{9}
\end{equation*}
$$

\]

with $d_{0}$ and $k_{0}$ given. The first order conditions are given by the following two difference equations of order two:

$$
\begin{align*}
u_{d}\left(T\left(k_{t}, k_{t+1}\right)-d_{t}, B d_{t+1}\right) B-\beta u_{c}\left(T\left(k_{t+1}, k_{t+2}\right)-d_{t+1}, B d_{t+2}\right) & =0 \\
u_{c}\left(T\left(k_{t}, k_{t+1}\right)-d_{t}, B d_{t+1}\right) T_{y}\left(k_{t}, k_{t+1}\right) & +  \tag{10}\\
\beta u_{c}\left(T\left(k_{t+1}, k_{t+2}\right)-d_{t+1}, B d_{t+2}\right) T_{k}\left(k_{t+1}, k_{t+2}\right) & =0
\end{align*}
$$

### 2.4 Steady state

A steady state is defined as $k_{t}=k^{*}, d_{t}=d^{*}$ for all $t$ solutions of the following equations

$$
\begin{align*}
\frac{u_{d}(T(k, k)-d, B d) B}{u_{c}(T(k, k)-d, B d)} & =\beta \\
-\frac{T_{y}(k, k)}{T_{k}(k, k)} & =\beta \tag{11}
\end{align*}
$$

Beside discussing the existence and uniqueness of the steady state, we need also to use the normalization parameter $B$ in order to normalize the stationary consumption $d$, rendering it constant when the discount factor $\beta$ is modified. As in the standard two-sector model, we get the following result:

Proposition 1. Under Assumptions 1-3, there exists a unique steady state $\left(k^{*}, d^{*}\right)$ solution of equations (11). Moreover, there exists a unique value $B^{*}$ such when $B=B^{*}$, the stationary consumption $d^{*}$ can be normalized to any value $\bar{d} \in\left(0, T\left(k^{*}, k^{*}\right)\right)$.

Proof. See Appendix 7.1.
A pair ( $k^{*}, d^{*}$ ) will be called the Modified Golden Rule. The stationary consumption of young agents is obtained from $c^{*}=T\left(k^{*}, k^{*}\right)-d^{*}$.

### 2.5 Characteristic polynomial

Based on the above computations, the characteristic polynomial is derived from total differentiation of equations (10). Denoting $T_{k}\left(k^{*}, k^{*}\right)=T_{k}^{*}$ and $T_{k k}\left(k^{*}, k^{*}\right)=T_{k k}^{*}$, we get:

[^2]Lemma 1. Under Assumptions 1-3, the degree-4 characteristic polynomial is given by

$$
\begin{align*}
\mathcal{P}(\lambda) & =\left[\lambda^{2}-\lambda\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{1}{\beta}\right] \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \\
& -\lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\beta T_{k}^{* 2}}{b \epsilon_{c c} c^{*} T_{k k}^{*}}\left(\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}\right) \tag{12}
\end{align*}
$$

Proof. See Appendix 7.2.
Remark: Notice that if $b=0$, we get the one-sector formulation with a two-dimensional dynamical system as considered in Michel and Venditti [13]. The characteristic polynomial can indeed be simplified as follows

The same conclusions as in Michel and Venditti [13] are obviously derived.
Similarly, if the utility function is additively separable, i.e. $u_{c d}=u_{d c}=0$, we get the two-sector optimal growth formulation with a two-dimensional dynamical system as considered in Benhabib and Nishimura [6]. The characteristic polynomial can indeed be simplified as follows

$$
\mathcal{P}(\lambda)=\lambda^{2}-\lambda(1+\beta) \frac{\frac{\beta T_{k}^{* 2}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-\left(\beta+b^{2}\right)}{\frac{\beta T_{k}^{*}}{\epsilon_{c c}^{*} c^{*} T_{k k}^{*}}-(1+\beta) b}+\frac{1}{\beta}
$$

The same conclusions as in Benhabib and Nishimura [6] are then derived.
Under Assumption 2, the sign of the expression $\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}$ is given by the sign of the cross derivative $u_{c d}$, i.e. by the opposite of the sign of $\epsilon_{c d}, \epsilon_{d c}$, which is a crucial ingredient to determine the local stability properties of the steady state. However, a degree-4 polynomial remains quite difficult to analyze. But we easily notice from (12) that if the utility function is nonstrictly concave, i.e. if $\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}=0$, then the degree-4 polynomial simplifies to a product of two degree-2 polynomials which are then quite simple to solve. We therefore introduce the following Assumption:

Assumption 4. The utility function $u(c, B d)$ is concave non-strictly, i.e. $\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}=0$.

## 3 Period-two cycles under non-strictly concave preferences

As a preliminary result, we show that under the additional Assumption 4, the characteristic roots cannot be complex

Lemma 2. Under Assumptions 1-4, the characteristic roots are real.

## Proof. See Appendix 7.3.

Following simultaneously the same methodologies as in the two-sector optimal growth model and the optimal growth solution of the aggregate OLG model, we discuss the local stability properties of equilibrium paths depending both on the sign of the capital intensity difference across sectors $b$ and the sign of the cross derivative $u_{c d}$, i.e. of the two elasticities $\epsilon_{c d}$ and $\epsilon_{d c}$.

We first provide with the following Proposition some simple conditions ensuring the saddle-point property.

Proposition 2. Under Assumptions 1-4, if $b \geq 0$ and $\epsilon_{c d}, \epsilon_{d c} \geq 0$, i.e. $u_{c d} \leq 0$, then the equilibrium path is monotone and the steady-state $\left(k^{*}, d^{*}\right)$ is a saddle-point.

Proof. See Appendix 7.4.
We now show that competitive equilibrium cycles may occur under a quite large set of circumstances.

Proposition 3. Under Assumptions 1-4, the following results hold:
i) When the investment good is capital intensive, i.e. $b \geq 0$, let $\epsilon_{c d}, \epsilon_{d c}<$ 0 , i.e. $u_{c d}>0$. Then the steady state $\left(k^{*}, d^{*}\right)$ is saddle-point stable with damped oscillations if and only if $\epsilon_{c c} \in\left(0,-\epsilon_{d c}\right) \cup\left(-\epsilon_{d c} / \beta,+\infty\right)$. Moreover, when $\epsilon_{c c}$ crosses the bifurcation values $-\epsilon_{d c}$ or $-\epsilon_{d c} / \beta,\left(k^{*}, d^{*}\right)$ undergoes a flip bifurcation leading to persistent period-2 cycles.
ii) When $\epsilon_{c d}, \epsilon_{d c} \geq 0$, i.e. $u_{c d} \leq 0$, let the consumption good be capital intensive, i.e. $b<0$. Then the steady state $\left(k^{*}, d^{*}\right)$ is saddle-point stable with damped oscillations if and only if $b \in(-\infty,-1) \cup(-\beta, 0)$. Moreover, if there is some $\beta^{*} \in(0,1)$ such that $b \in\left(-\infty,-\beta^{*}\right)$, then there exists $\bar{\beta} \in(0,1)$ such that, when $\beta$ crosses $\bar{\beta}$ from above, $\left(k^{*}, d^{*}\right)$ undergoes a flip bifurcation leading to persistent period-2 cycles.
iii) When the consumption good is capital intensive, i.e. $b<0$, and $\epsilon_{c d}, \epsilon_{d c}<0$, i.e. $u_{c d}>0$, the steady state $\left(k^{*}, d^{*}\right)$ is saddle-point stable with damped oscillations if and only if $b \in(-\infty,-1) \cup(-\beta, 0)$ and $\epsilon_{c c} \in$ $\left(0,-\epsilon_{d c}\right) \cup\left(-\epsilon_{d c} / \beta,+\infty\right)$. Moreover, if there is some $\beta^{*} \in(0,1)$ such that $b \in\left(-1,-\beta^{*}\right)$, then there exists $\bar{\beta} \in(0,1)$ such that, when $\beta$ crosses $\bar{\beta}$ from above or $\epsilon_{c c}$ crosses the bifurcation values $-\epsilon_{d c}$ or $-\epsilon_{d c} / \beta$, $\left(k^{*}, d^{*}\right)$ undergoes a flip bifurcation leading to persistent period-2 cycles.

Proof. See Appendix 7.5.

Proposition 3 provides two independent mechanisms leading to the existence of endogenous fluctuations. The first one is based on the properties of preferences through the sign of the cross derivative $u_{c d}$ and is the more interesting as it allows to generate period-2 cycles in a two-sector model even under a capital intensive investment good sector, a condition which is known since Benhabib and Nishimura [7] to guarantee monotone convergence in a standard optimal growth model. In order to provide an economic intuition, let us consider an instantaneous increase in the capital stock $k_{t}$. From the equality $c_{t}+d_{t}=T\left(k_{t}, y_{t}\right)$ and the fact that $T_{k}>0$, we derive that $c_{t}$ increases, and thus, using the fact that the marginal utility of second period consumption $u_{d}$ is larger as $u_{d c}>0$, a constant utility level $u\left(c_{t}, d_{t+1}\right)$ can be obtained from a decrease of $d_{t+1}$. Consider then the first equation in (10). We derive for a given $d_{t+2}$

$$
\frac{\Delta c_{t+1}}{\Delta c_{t}}=\frac{u_{d c}}{u_{c c} \beta}+\frac{u_{d d}}{u_{c c} \beta} \frac{\Delta d_{t+1}}{\Delta c_{t}}<0
$$

It follows therefore from the equality $c_{t+1}+d_{t+1}=T\left(k_{t+1}, y_{t+1}\right)$ that total consumption at time $t+1$ is lower, implying for a constant $y_{t+1}$, a lower capital stock $k_{t+1}$. Endogenous fluctuations are thus generated from consumption intertemporal allocations.

The second mechanism is, as in the two-sector optimal growth model, based on the properties of sectoral technologies through the sign of the capital intensity difference across sectors. Following Benhabib and Nishimura [7], we can use the Rybczinski and Stolper-Samuelson effects to provide a simple economic intuition for this result. Assume indeed that the consumption good is capital intensive, i.e. $b<0$, and consider an instantaneous increase in the capital stock $k_{t}$. This results in two opposing forces:

- The trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease of the output of the capital good $y_{t}$. This tends to lower the investment and the capital stock in the next period $k_{t+1}$.
- In the next period the decrease of $k_{t+1}$ implies again through the Rybczinsky effect an increase of the output of the capital good $y_{t+1}$. Indeed the decrease of $k_{t+1}$ improves the trade-off in production in favor of the investment good which is relatively less intensive in capital and this tends to increase the investment and the capital stock in period $t+2, k_{t+2}$.

Of course, under both mechanisms, the existence of persistent fluctuations require that the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. A minimum level of myopia, i.e. a low enough value for the discount rate $\beta$, is thus necessary.

Note finally that in case iii) of Proposition 3, both mechanisms hold at the same time.

As a simple illustration, consider the case of a linear homogeneous utility function. Both consumption levels are then normal goods and the concavity is non-strict. Building on the homogeneity of degree 1 , we introduce the share of first period consumption within total utility $\phi(c, d) \in(0,1)$ defined as follows:

$$
\begin{equation*}
\phi(c, B d)=\frac{u_{c}(c, B d) c}{u(c, B d)} \tag{13}
\end{equation*}
$$

The share of second period consumption within total utility is similarly defined as $1-\phi(c, B d) \in(0,1)$. Moreover, the cross derivative $u_{c d}$ is obviously positive. ${ }^{4}$ Focusing on the more interesting case where endogenous fluctuations arise under a capital intensive investment good, we get the following Corollary:

Corollary 1. Under Assumption 1, let the investment good be capital intensive, i.e. $b \geq 0$, and the utility function $u(c, B d)$ be linear homogeneous. Then the steady state $\left(k^{*}, d^{*}\right)$ is saddle-point stable with damped oscillations if and only if $\phi \in(0, \underline{\phi}) \cup(\bar{\phi},+\infty)$, with $\underline{\phi}=1 / 2$ and $\bar{\phi}=1 /(1+\beta)$. Moreover, when $\phi$ crosses $\overline{\text { the }}$ bifurcation values $\phi$ or $\bar{\phi},\left(k^{*}, d^{*}\right)$ undergoes a flip bifurcation leading to persistent period-2 cycles.

Proof. See Appendix 7.6.
A particular example of a linear homogeneous utility function is provided by the CES formulation such that:

$$
\begin{equation*}
u(c, B d)=\left[\theta c^{-\rho}+(1-\theta) d^{-\rho}\right]^{-1 / \rho} \tag{14}
\end{equation*}
$$

with $B=1,{ }^{5} \theta \in(0,1), \rho>-1$, and where

$$
\begin{equation*}
\phi(c, d)=\frac{\theta c^{-\rho}}{\theta c^{-\rho}+(1-\theta) d^{-\rho}} \tag{15}
\end{equation*}
$$

The ratio $1 /(1+\rho)$ provides the elasticity of substitution between $c$ and $d$. With such a formulation, we easily derive

$$
\frac{c}{d}=\frac{\beta \phi}{1-\phi}=\left(\frac{\beta \theta}{1-\theta}\right)^{\frac{1}{1+\rho}}
$$

and considering given values for $k^{*}$ and $T\left(k^{*}, k^{*}\right)$, the steady state $d^{*}$ is given by

[^3]\[

$$
\begin{equation*}
d^{*}=\frac{(1-\phi) T\left(k^{*}, k^{*}\right)}{1-\phi(1-\beta)}=\frac{T\left(k^{*}, k^{*}\right)}{1+\left(\frac{\theta \beta}{1-\theta}\right)^{\frac{1}{1+\rho}}} \in\left(0, T\left(k^{*}, k^{*}\right)\right) \tag{16}
\end{equation*}
$$

\]

and obviously

$$
\begin{equation*}
c^{*}=T\left(k^{*}, k^{*}\right)-d^{*}=\frac{\beta \phi T\left(k^{*}, k^{*}\right)}{1-\phi(1-\beta)}=\frac{\left(\frac{\theta \beta}{1-\theta}\right)^{\frac{1}{1+\rho}} T\left(k^{*}, k^{*}\right)}{1+\left(\frac{\theta \beta}{1-\theta}\right)^{\frac{1}{1+\rho}}} \in\left(0, T\left(k^{*}, k^{*}\right)\right) \tag{17}
\end{equation*}
$$

Plugging these expressions into (15) gives

$$
\begin{equation*}
\phi\left(c^{*}, d^{*}\right)=\frac{1}{1+\beta^{\frac{\rho}{1+\rho}}\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{1+\rho}}} \equiv \phi(\theta) \tag{18}
\end{equation*}
$$

and we easily get

$$
\begin{equation*}
\lim _{\theta \rightarrow 1} \phi(\theta)=1 \text { and } \lim _{\theta \rightarrow 0} \phi(\theta)=0 \tag{19}
\end{equation*}
$$

It follows therefore that there exist $1>\bar{\theta}>\underline{\theta}>0$ such that $\underline{\phi}=\phi(\underline{\theta})$, $\bar{\phi}=\phi(\bar{\theta})$ with

$$
\begin{equation*}
\bar{\theta}=\frac{1}{1+\beta} \text { and } \underline{\theta}=\frac{1}{1+\beta^{-\rho}} \tag{20}
\end{equation*}
$$

and the results of Corollary 1 hold.

## 4 Quasi-periodic cycles under strictly concave preferences

Up to now we have simplified the analysis to the consideration of a nonstrictly concave utility function in order to reduce the degree- 4 characteristic polynomial to the product of two degree-2 polynomials. An illustration of such utility function has been provided by a CES linear homogenous specification as given by (21). In such a framework, we have shown that the characteristic roots are necessarily real and that endogenous fluctuations can occur through the existence of period-two cycles. But from an empirical point of view, period-two cycles are associated to the unrealistic property of negative auto-correlation of variables. In order to solve this problem, we need to focus on the existence of complex characteristic roots with which quasi-periodic cycles occurring through a Hopf bifurcation can generate fluctuations that are compatible with positive auto-correlations. Such a property is then required to provide an empirically relevant description of long-run fluctuations of variables such as bequests.

We can start by providing general sufficient conditions allowing to rule out the existence of complex roots.

Proposition 4. Under Assumptions 1-3, let the utility function $u(c, B d)$ is be strictly concave. Then the roots of the characteristic polynomial (12) are necessarily real in the following cases:
i) for any sign of $\epsilon_{c d}, \epsilon_{c d}$ if the investment good sector is capital intensive, i.e. $b>0$,
ii) if $\epsilon_{c d}, \epsilon_{c d}>0$ and the consumption good sector is capital intensive, i.e. $b<0$.

Proof. See Appendix 7.7.
Necessary conditions for the existence of complex roots are therefore based on the two mechanisms that generate endogenous fluctuations in the non-strictly concave case, namely $b<0$ and $\epsilon_{c d}, \epsilon_{c d}<0$. In order to study whether complex characteristic roots and a Hopf bifurcation with quasiperiodic cycles can occur, let us consider now a particular class of strictly concave utility function as given by the following generalized CES formulation

$$
\begin{equation*}
u(c, B d)=\left[\theta c^{-\rho}+(1-\theta) d^{-\rho}\right]^{-\gamma / \rho} \tag{21}
\end{equation*}
$$

with $B=1, \theta \in(0,1), \rho>-1$ and $\gamma \in(0,1]$. Here $\gamma$ is the degree of homogeneity of $u(c, B d)$ which is thus strictly concave if $\gamma<1$. We then get the share of first period consumption within total utility

$$
\begin{equation*}
\phi(c, d)=\frac{\gamma \theta c^{-\rho}}{\theta c^{-\rho}+(1-\theta) d^{-\rho}} \in(0, \gamma) \tag{22}
\end{equation*}
$$

while the share of second period consumption within total utility is now given by $\gamma-\phi(c, d) .{ }^{6}$ We then easily derive from the first order condition $u_{d} B=\beta u_{c}$ and the fact that $c^{*}=T\left(k^{*}, k^{*}\right)-d^{*}$

$$
\frac{c}{d}=\frac{\beta \phi}{\gamma-\phi}=\left(\frac{\beta \theta}{1-\theta}\right)^{\frac{1}{1+\rho}}
$$

As in the case $\gamma=1$, considering given values for $k^{*}$ and $T\left(k^{*}, k^{*}\right)$, the steady state $d^{*}$ is given by

$$
\begin{equation*}
d^{*}=\frac{(\gamma-\phi) T\left(k^{*}, k^{*}\right)}{\gamma-\phi(1-\beta)}=\frac{T\left(k^{*}, k^{*}\right)}{1+\left(\frac{\frac{\theta}{1}}{1-\theta}\right)^{\frac{1}{1+\rho}}} \in\left(0, T\left(k^{*}, k^{*}\right)\right) \tag{23}
\end{equation*}
$$

and obviously

$$
\begin{equation*}
c^{*}=T\left(k^{*}, k^{*}\right)-d^{*}=\frac{\beta \phi T\left(k^{*}, k^{*}\right)}{\gamma-\phi(1-\beta)}=\frac{\left(\frac{\theta \beta}{1-\theta}\right)^{\frac{1}{1+\rho}} T\left(k^{*}, k^{*}\right)}{1+\left(\frac{\theta \beta}{1-\theta}\right)^{\frac{1}{1+\rho}}} \in\left(0, T\left(k^{*}, k^{*}\right)\right) \tag{24}
\end{equation*}
$$

Moreover from (5)-(6) we get

$$
\epsilon_{c d}=-\frac{\epsilon_{c c}}{1-\epsilon_{c c}(1-\gamma)}, \quad \epsilon_{d c}=-\frac{(\gamma-\phi) \epsilon_{c c}}{\phi\left[1-\epsilon_{c c}(1-\gamma)\right]}, \quad \epsilon_{d d}=\frac{(\gamma-\phi) \epsilon_{c c}}{\phi-\epsilon_{c c}(1-\gamma)(2 \phi-\gamma)}
$$

It is important to note here that, when expressed in terms of these elasticities, the concavity of the utility function requires the following restriction:

[^4]Assumption 5. $\epsilon_{c c}<\bar{\epsilon}_{c c} \equiv \frac{\gamma}{\phi(1-\gamma)}$
Under this restriction, we obviously get $\epsilon_{d d}>0$ while $\epsilon_{c d}, \epsilon_{c d}<0$ if and only if $\epsilon_{c c}<1 /(1-\gamma) \equiv \tilde{\epsilon}_{c c} \in\left(0, \bar{\epsilon}_{c c}\right)$.

Assume now as in Baierl et al. [2] that the consumption and investment goods are produced with Cobb-Douglas technologies as follows

$$
\begin{equation*}
y_{0}=k_{0}^{\alpha_{0}} l_{0}^{1-\alpha_{0}}, y=k_{1}^{\alpha_{1}} l_{1}^{1-\alpha_{1}} \tag{25}
\end{equation*}
$$

As detailed in Appendix 7.8, we can show that

$$
\begin{align*}
k^{*} & =\frac{\alpha_{0}\left(1-\alpha_{1}\right)\left(\beta \alpha_{1}\right)^{\frac{1}{1-\alpha_{1}}}}{\alpha_{1}\left(1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)\right]} \\
T\left(k^{*}, k^{*}\right) & =\left(\frac{\alpha_{0}\left(1-\alpha_{1}\right)}{\left(1-\alpha_{0}\right) \alpha_{1}}\right)^{\alpha_{0}} \frac{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\beta \alpha_{1}\right)^{\frac{\alpha_{0}}{1-\alpha_{1}}}}{1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)} \\
r^{*}=T_{k}\left(k^{*}, k^{*}\right) & =\alpha_{0}\left(\frac{\left(1-\alpha_{0}\right) \alpha_{1}}{\alpha_{0}\left(1-\alpha_{1}\right)}\right)^{1-\alpha_{0}}\left(\beta \alpha_{1}\right)^{-\frac{1-\alpha_{0}}{1-\alpha_{1}}}  \tag{26}\\
T_{k k}\left(k^{*}, k^{*}\right) & =-\frac{T_{k}\left(k^{*}, k^{*}\right)}{k^{*}} \frac{\left(1-\alpha_{0}\right)^{2}}{1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)} \\
b & =\frac{\beta\left(\alpha_{1}-\alpha_{0}\right)}{1-\alpha_{0}}
\end{align*}
$$

The degree- 4 characteristic polynomial as given by Lemma 1 becomes here

$$
\begin{align*}
\mathcal{P}(\lambda) & =\left[\lambda^{2}+\lambda\left(\frac{(\gamma-\phi)^{2}+\beta \phi^{2}-\beta \phi \epsilon_{c c}(1-\gamma)(2 \phi-\gamma)}{\beta \phi(\gamma-\phi)\left[1-\epsilon_{c c}(1-\gamma]\right.}\right)+\frac{1}{\beta}\right] \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \\
& -\lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]}{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)} \frac{\epsilon_{c c}(1-\gamma)\left[\gamma-\epsilon_{c c} \phi(1-\gamma)\right]}{(\gamma-\phi)\left[1-\epsilon_{c c}(1-\gamma)\right]} \tag{27}
\end{align*}
$$

We first provide sufficient conditions to ensure saddle-point property of the steady state with real characteristic roots.
Proposition 5. Let the utility function and the sectoral production functions be given respectively by (21) and (25), and assume that $\epsilon_{c c}<1 /(1-\gamma)$ and $\alpha_{0}>\alpha_{1}$ such that $b \in(-\infty,-1) \cup(-\beta, 0)$. Then there exist $0<\underline{\phi}<$ $\bar{\phi}<\gamma$ and $\hat{\epsilon}_{c c} \in\left(0, \tilde{\epsilon}_{c c}\right)$ such that when $\phi \in(0, \phi) \cup(\bar{\phi}, \gamma)$ the characteristic roots are real and the steady-state is saddle-point stable. Moreover,
i) when $\phi \in(\bar{\phi}, \gamma)$, the optimal path converges towards the steady state with oscillations if $\epsilon_{c c} \in\left(0, \hat{\epsilon}_{c c}\right)$ or monotonically if $\epsilon_{c c} \in\left(\hat{\epsilon}_{c c}, \tilde{\epsilon}_{c c}\right)$,
ii) when $\phi \in(0, \underline{\phi})$, the optimal path converges towards the steady state with oscillations.

Proof. See Appendix 7.9.
Proposition 5 implies that the existence of complex roots, if any, requires to consider values of $\phi$ such that $\phi \in(\underline{\phi}, \bar{\phi})$. As we do not have any sufficient
conditions to get complex characteristic roots that could have a modulus equal to one, let us focus on a numerical illustration with $b<0$ and $\epsilon_{c d}, \epsilon_{c d}<$ 0 . Considering that the annual discount factor is often estimated to be around 0.96 and that one period in an OLG model is about 30 years, we consider here that $\beta=0.96^{30} \approx 0.294$. Focusing on a slight deviation with respect to the linear homogeneous case with $\gamma=0.98$, let us then assume a standard value $\epsilon_{c c}=1$ that satisfies $\epsilon_{c c}<\bar{\epsilon}_{c c}$. We also consider $\alpha_{0}=0.6$ and $\alpha_{1}=0.21$ so that the consumption good is capital intensive with $b \in$ $(-\beta, 0)$. The bounds exhibited in Proposition 5 are equal to $\phi \approx 0.38858$ and $\bar{\phi} \approx 0.865$. We then find that the characteristic polynomial $(27)$ admits four characteristic roots $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ that are complex conjugate by pair with $\lambda_{1} \lambda_{2}>1$ and $\lambda_{3} \lambda_{4}<1$ if $\phi \in\left(\phi, \phi^{H}\right) \cup\left(\bar{\phi}^{H}, \bar{\phi}\right)$ while $\lambda_{3} \lambda_{4}>1$ if $\phi \in\left(\underline{\phi}^{H}, \bar{\phi}^{H}\right)$, with $\underline{\phi}^{H} \equiv 0.5674$ and $\overline{\bar{\phi}}^{\bar{H}} \equiv 0.6713$. Moreover $\lambda_{3} \lambda_{4}=1$ when $\bar{\phi}=\phi^{H}$ or $\bar{\phi}^{H}$. As a result $\underline{\phi}^{H}$ and $\bar{\phi}^{H}$ are Hopf bifurcation values giving rise to quasi-periodic cycles in their neighborhood.

We can then derive by continuity:
Proposition 6. Let the utility function and the sectoral production functions be given respectively by (21) and (25), and assume that $\epsilon_{c c}<1 /(1-\gamma)$ and $\alpha_{0}>\alpha_{1}$ such that $b \in(-\infty,-1) \cup(-\beta, 0)$. Then there exist an open set of values for $\left(\beta, \alpha_{0}, \alpha_{1}, \gamma\right)$ with $\gamma<1$ and two critical values $\phi^{H}, \bar{\phi}^{H} \in(\phi, \bar{\phi})$ such that the steady state $\left(k^{*}, d^{*}\right)$ is saddle-point stable with damped oscillations if $\phi \in\left(\phi, \phi^{H}\right) \cup\left(\bar{\phi}^{H}, \bar{\phi}\right)$. Moreover, when $\phi$ crosses the bifurcation values $\underline{\phi}^{H}$ or $\bar{\phi} \overline{\bar{H}},\left(k^{*}, d^{*}\right)$ undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles.

From a theoretical point of view, Proposition 6 provides a strong conclusion as it shows that a Hopf bifurcation and quasi-periodic cycles can occur in a two-sector optimal growth framework as long as it is based on an OLG structure with non-separable and strictly concave preferences. Such a result is drastically different from what can be obtained in standard optimal growth models as the existence of complex roots requires to consider at least three sectors. ${ }^{7}$ From an empirical point of view, Proposition 6 also provides a strong conclusion which is related to the quasi periodicity of the cycles leading to positive auto-correlations of variables. Such a property is then required to provide an empirically relevant description of long-run fluctuations of bequests. We need however to show now that the existence of optimal endogenous cycles is compatible with strictly positive bequest transmissions across generations.

[^5]
## 5 The solution with altruistic agents and a bequest motive

Let us consider now a decentralized economy composed of overlapping generations of parents loving their children. As in the Barro [3] formulation, each agent is altruistic towards his descendant through a bequest motive. Parents indeed care about their child's welfare by taking into account their child's utility into their own utility function. They are now price-takers, considering as given the prices $p_{t}, w_{t}$ and $r_{t+1}$ as defined by (2) and (3), and determine their optimal decisions with respect to their budget constraints

$$
\begin{equation*}
w_{t}+p_{t} x_{t}=c_{t}+s_{t} \text { and } R_{t+1} s_{t}=d_{t+1}+p_{t+1} x_{t+1} \tag{28}
\end{equation*}
$$

with $R_{t+1}=r_{t+1} / p_{t}$ the gross rate of return, $s_{t}$ the savings of young agents born in $t$ and $x_{t}$ the amount of bequest transmitted at time $t$ by agents born in $t-1$. Note that bequest $x_{t}$ is expressed as an investment good and requires the relative price $p_{t}$ to enter the budget constraints. In each period, bequests must be non-negative:

$$
\begin{equation*}
x_{t} \geq 0 \text { for all } t \geq 0 \tag{29}
\end{equation*}
$$

An altruistic agent has a utility function given by the following Bellman equation

$$
\begin{align*}
V_{t}\left(x_{t}\right) & =\max _{c_{t}, d_{t+1}, s_{t}, x_{t+1}}\left\{u\left(c_{t}, B d_{t+1}\right)+\beta V_{t+1}\left(x_{t+1}\right)\right\} \\
& =\max _{\left\{c_{t}, d_{t+1}, s_{t}, x_{t+1}\right\}} \sum_{t=0}^{+\infty} \beta^{t} u\left(c_{t}, B d_{t+1}\right) \tag{30}
\end{align*}
$$

subject to (28) and (29). $\beta$ is now interpreted as the intergenerational degree of altruism. It is well-known from the first welfare theorem that this altruistic problem is equivalent to the central planner problem (8), and the equilibrium is the unique Pareto optimum which coincides with the centralized solution. However, such an equivalence requires the non-negativity constraints of bequests (29) to hold with a strict inequality in order to preserve the link across generations.

Denoting $q_{t}$ the shadow price of bequest $x_{t}$, we define the generalized Lagrangian associated to the optimization program (30)

$$
\mathcal{L}=u\left(c_{t}, B d_{t+1}\right)+\beta \frac{q_{t+1}}{p_{t+1}}\left[R_{t+1}\left(w_{t}+p_{t} x_{t}-c_{t}\right)-d_{t+1}\right]-q_{t} x_{t}
$$

The first order conditions are the following

$$
\begin{aligned}
u_{c}\left(c_{t}, B d_{t+1}\right) & =\frac{\beta q_{t+1} R_{t+1}}{p_{t+1}} \\
u_{d}\left(c_{t}, B d_{t+1}\right) B & =\frac{\beta q_{t+1}}{p_{t+1}} \\
\frac{\beta q_{t+1} R_{t+1} p_{t}}{p_{t+1}} & \leq q_{t} \text { with an equality if } x_{t}>0
\end{aligned}
$$

Consider now the two budget constraints in (28) evaluated at the steady state. Solving with respect to $s_{t}$ using the fact that $s_{t}=p_{t} y_{t}=p_{t} k_{t+1}$ and $R_{t+1}=r_{t+1} / p_{t}$ we get

$$
\begin{align*}
p^{*} x^{*}\left(1-\frac{1}{R^{*}}\right) & =c^{*}+\frac{d^{*}}{R^{*}}-w^{*}=T\left(k^{*}, k^{*}\right)-w^{*}-d^{*}\left(1-\frac{1}{R^{*}}\right) \\
& =\left(r^{*} k^{*}-d^{*}\right)\left(1-\frac{1}{R^{*}}\right) \tag{31}
\end{align*}
$$

If $x^{*}>0$, i.e. $r^{*} k^{*}>d^{*}$, then we derive from the fist order conditions that $R^{*}=r^{*} / p^{*}=\beta^{-1}$ and $u_{d}\left(c^{*}, B d^{*}\right)=\beta u_{c}\left(c^{*}, B d^{*}\right)$, which are exactly the same conditions as (11). We then obtain:

Proposition 7. Under Assumptions 1-4, for any $\beta \in(0,1)$, there exists a unique value $B^{*}$ such when $B=B^{*}$, bequests are positive in the economy with degree of altruism equal to $\beta$.

Proof. See Appendix 7.10.
When bequests are positive at the steady state, then by continuity there are positive in a neighborhood of the steady state and the local stability properties provided in Propositions 2, 3 and 6 hold. In particular, the existence of optimal cycles and business fluctuations hold under positive bequests.

In order to illustrate this result, let us consider first the example of a linear homogeneous CES utility function as given by (21) with a CobbDouglas production structure as given by (25). Using (16) and (26), we derive that $r^{*} k^{*}>d^{*}$ and thus $x^{*}>0$ if and only if

$$
\alpha_{0}\left(\frac{\beta \theta}{1-\theta}\right)^{\frac{1}{1+\rho}}-\left(1-\alpha_{0}-\beta \alpha_{1}\right)>0
$$

It follows immediately that if $\alpha_{1}>1-\alpha_{0}$ and $\beta>\left(1-\alpha_{0}\right) / \alpha_{1}$, then $1-\alpha_{0}-\beta \alpha_{1}<0$ and $x^{*}>0$ for any $\theta \in(0,1)$. The existence of periodic cycles is thus compatible with positive bequests. Similarly, when $\alpha_{1}<1-\alpha_{0}$, straightforward computations show that $x^{*}>0$ if and only if

$$
\theta>\frac{1}{1+\left(\frac{\alpha_{0}}{1-\alpha_{0}-\beta \alpha_{1}}\right)^{1+\rho}{ }_{\beta}} \equiv \tilde{\theta}
$$

Considering the bounds (20), it follows that the conditions of Corollary 1 for the existence of period-2 cycles can be satisfied if $\tilde{\theta}<\underline{\theta}$. Sufficient conditions
for this inequality to be satisfied are given by $\alpha_{1} \in\left(1-2 \alpha_{0}, 1-\alpha_{0}\right)$ and $\beta>\left(1-\alpha_{0}\right) /\left(\alpha_{0}+\alpha_{1}\right) \equiv \underline{\beta}$ with $\underline{\beta}<1$. This example clearly shows that when the degree of altruism is large enough, endogenous optimal fluctuations are compatible with positive bequests. Moreover, this result holds for any sign of the capital intensity difference across sectors.

It is worth noticing that if, under $\alpha_{1} \in\left(1-2 \alpha_{0}, 1-\alpha_{0}\right)$, we assume that $\theta>\tilde{\theta}$ with $\tilde{\theta}>\bar{\theta}$, then bequests are positive but the conditions of Corollary 1 for the existence of period- 2 cycles cannot be satisfied and the steady state is saddle-point stable. This inequality is satisfied if and only if $\alpha_{1} \in\left(1-2 \alpha_{0}, 1-\alpha_{0}\right), \alpha_{0}<1 / 2$ and $\beta<\left(1-2 \alpha_{0}\right) / \alpha_{1}$. Therefore, if the degree of altruism is not large enough, persistent endogenous fluctuations cannot arise.

Let us finally illustrate the possible existence of quasi-periodic cycles under positive bequests when the utility function is strictly concave as in Section 4. Considering the formulation of Proposition 6, we provide here conditions for a positive bequest in terms of the parameter $\phi$. Using (23) and (26), we derive that $r^{*} k^{*}>d^{*}$ and thus $x^{*}>0$ if and only if

$$
\alpha_{0} \phi \beta-(\gamma-\phi)\left(1-\alpha_{0}-\beta \alpha_{1}\right)>0
$$

Consider then the particular illustration in Section 4 which is such that $1-\alpha_{0}-\beta \alpha_{1}>0$ and $\alpha_{0}>\alpha_{1}$. It follows that bequests are positive if and only if

$$
\phi>\frac{\gamma\left(1-\alpha_{0}-\beta \alpha_{1}\right)}{1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)} \equiv \tilde{\phi}
$$

With $\gamma=0.98, \alpha_{0}=0.6$ and $\alpha_{1}=0.2$, we get $\tilde{\phi} \approx 0.646 \in\left(\phi^{H}, \bar{\phi}^{H}\right)$. It follows that positive bequests are compatible with quasi-periodic cycles. Indeed, the steady state, which is characterized by strictly positive bequests if $\phi>\tilde{\phi}$, is saddle-point stable with damped oscillations if and only if $\phi \in\left(\bar{\phi}^{H}, \bar{\phi}\right)$. Moreover, when $\phi$ crosses the bifurcation values $\bar{\phi}^{H}$ from above, the steady state undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles.

## 6 Concluding comments

## 7 Appendix

### 7.1 Proof of Proposition 1

Consider in a first step the second equation in (11). Notice that the steady state value for $k$ only depends on the characteristics of the technologies
and is independent from the utility function. Moreover, this equation is equivalent to the equation which defines the stationary capital stock of a standard two-sector optimal growth model. The proof of Theorem 3.1 in Becker and Tsyganov [4] restricted to the case of one homogeneous agent applies so that there exists one unique $k^{*}$ solution of this equation.

Consider now the first equation in (11) evaluated at $k^{*}$. We get:

$$
\begin{equation*}
\frac{u_{d}\left(T\left(k^{*}, k^{*}\right)-d, B d\right) B}{u_{c}\left(T\left(k^{*}, k^{*}\right)-d, B d\right)} \equiv h(d)=\beta \tag{32}
\end{equation*}
$$

The function $h(d)$ is defined over $\left(0, T\left(k^{*}, k^{*}\right)\right)$ and satisfies

$$
h^{\prime}(d)=\frac{\frac{B u_{d d}}{u_{d}}-\frac{u_{c d}}{u_{c}}+\frac{u_{c c}}{u_{c}}-\frac{B u_{c d}}{u_{d}}}{u_{c} u_{d}}=-\beta\left[\frac{1}{d}\left(\frac{1}{\epsilon_{d d}}-\frac{1}{\epsilon_{c d}}\right)+\frac{1}{c}\left(\frac{1}{\epsilon_{c c}}-\frac{1}{\epsilon_{d c}}\right)\right]
$$

Assumption 3 implies that $h^{\prime}(d)<0$. This monotonicity property together with the boundary conditions in Assumption 2 finally ensure the existence and uniqueness of a solution $d^{*} \in\left(0, T\left(k^{*}, k^{*}\right)\right)$ of equation (32).

For a given $k^{*}$, consider a particular value $d^{*}=\bar{d} \in\left(0, T\left(k^{*}, k^{*}\right)\right) . \bar{d}$ is a steady state if

$$
\begin{equation*}
\frac{u_{d}\left(T\left(k^{*}, k^{*}\right)-\bar{d}, B \bar{d}\right) B}{u_{c}\left(T\left(k^{*}, k^{*}\right)-\bar{d}, B \bar{d}\right)} \equiv g(B)=\beta \tag{33}
\end{equation*}
$$

We easily get

$$
g^{\prime}(B)=-\frac{u_{d}}{u_{c}}\left[\frac{1}{\epsilon_{d d}}-\frac{1}{\epsilon_{c d}}-1\right]
$$

which is generically different from zero. Therefore, under the boundary conditions in Assumption 2, there generically exists a unique value $B^{*}$ such that when $B=B^{*}, d^{*}=\bar{d}$ is a normalized steady state.

### 7.2 Proof of Lemma 1

Using (5)-(6) and the fact that at the steady state $-T_{y}^{*}=\beta T_{k}^{*}$, total differentiation of the first order equations (10) gives after tedious but straightforward computations:

$$
\begin{aligned}
& -\Delta k_{t} \frac{\beta T_{k}^{*} \epsilon_{c c}}{\epsilon_{d c}}+\Delta k_{t+1} \beta T_{k}^{*}\left(1+\frac{\beta \epsilon_{c c}}{\epsilon_{d c}}\right)+\Delta d_{t} \frac{\beta \epsilon_{c c}}{\epsilon_{d c}}-\Delta d_{t+1} \beta\left(1+\frac{\beta \epsilon_{c c} \epsilon_{c d}}{\epsilon_{d c} \epsilon_{d d}}\right) \\
= & \Delta k_{t+2} \beta^{2} T_{k}^{*}-\Delta d_{t+2} \frac{\beta^{2} \epsilon_{c c}}{\epsilon_{d c}} \\
& \Delta k_{t}\left(\frac{\beta T_{k}^{* 2}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-b\right)-\Delta k_{t+1}\left(\frac{\beta(1+\beta) T_{k}^{* 2}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-\Delta-b^{2}\right)-\Delta d_{t} \frac{\beta T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}} \\
+ & \Delta d_{t+1} \frac{\beta T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}}\left(1+\frac{\beta \epsilon_{c c}}{\epsilon_{d c}}\right)=-\Delta k_{t+2} \beta\left(\frac{\beta T_{k}^{* 2}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-b\right)+\Delta d_{t+2} \frac{\beta^{2} T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}}
\end{aligned}
$$

Denoting $\Delta \xi_{t}=\Delta k_{t+1}$ and $\Delta \zeta_{t}=\Delta d_{t+1}$, we get the following matrix expression of the previous linear system:

$$
\begin{aligned}
& \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \beta^{2} T_{k}^{*} & -\frac{\beta^{2} \epsilon_{c c}}{\epsilon_{d c}} \\
0 & 0 & -\left(\frac{\beta T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-b\right) & \frac{\beta^{2} T_{k}^{*}}{\epsilon_{c c} T_{k}^{*} T_{k k}^{*}}
\end{array}\right)\left(\begin{array}{c}
\Delta k_{t+1} \\
\Delta d_{t+1} \\
\Delta \xi_{t+1} \\
\Delta \zeta_{t+1}
\end{array}\right) \\
& =\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{\beta T_{k}^{*} \epsilon_{c c}}{\epsilon_{d c}} & \frac{\beta \epsilon_{c c}}{\epsilon_{d c}} & \beta T_{k}^{*}\left(1+\frac{\beta \epsilon_{c c}}{\epsilon_{d c}}\right) & -\beta\left(1+\frac{\beta \epsilon_{c c \epsilon_{c d}}}{\epsilon_{d c} \epsilon_{d d}}\right) \\
\frac{\beta T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}}-b & \frac{\beta T_{k}^{*}}{\epsilon_{c c} c^{*} T_{k k}^{*}} & -\frac{\beta(1+\beta) T_{k}^{* 2}}{\epsilon_{c c} c^{*} T_{k k}^{*}}+\beta+b^{2} & \frac{\beta T_{k}^{*}}{\epsilon_{c c c}^{*} T_{k k}^{*}}\left(1+\frac{\beta \epsilon_{c c}}{\epsilon_{d c}}\right)
\end{array}\right)\left(\begin{array}{c}
\Delta k_{t} \\
\Delta d_{t} \\
\Delta \xi_{t} \\
\Delta \zeta_{t}
\end{array}\right) \\
& \Leftrightarrow A\left(\begin{array}{c}
\Delta k_{t+1} \\
\Delta d_{t+1} \\
\Delta \xi_{t+1} \\
\Delta \zeta_{t+1}
\end{array}\right)=B\left(\begin{array}{c}
\Delta k_{t} \\
\Delta d_{t} \\
\Delta \xi_{t} \\
\Delta \zeta_{t}
\end{array}\right)
\end{aligned}
$$

with

$$
A=\left(\begin{array}{cc}
0 & I \\
0 & A_{22}
\end{array}\right) \text { and } B=\left(\begin{array}{cc}
0 & I \\
B_{21} & B_{22}
\end{array}\right)
$$

Matrix $A$ is invertible as

$$
\operatorname{det} A=\operatorname{det} A_{22}=\frac{\delta^{3} b \epsilon_{c c}}{\epsilon_{d c}}
$$

and we get

$$
A^{-1}=\left(\begin{array}{cc}
0 & I \\
0 & A_{22}^{-1}
\end{array}\right)=\left(\begin{array}{cc}
\frac{T_{c k}^{*}}{\beta b \epsilon_{c} T^{*} T_{k k}^{*}} & \frac{1}{\beta b} \\
\frac{\epsilon_{d c}}{\beta^{2} \epsilon_{c c}}\left(\frac{\beta T_{k}^{2} k}{b \epsilon_{c c} c^{*} T_{k k}^{*}}-1\right) & \frac{\epsilon_{d c} T_{k}^{*}}{\beta b c_{c c}}
\end{array}\right)
$$

The linearized dynamical system can then be expressed as follows

$$
\left(\begin{array}{c}
\Delta k_{t+1} \\
\Delta d_{t+1} \\
\Delta \xi_{t+1} \\
\Delta \zeta_{t+1}
\end{array}\right)=A^{-1} B\left(\begin{array}{c}
\Delta k_{t} \\
\Delta d_{t} \\
\Delta \xi_{t} \\
\Delta \zeta_{t}
\end{array}\right)=\left(\begin{array}{cc}
0 & I \\
A_{22}^{-1} B_{21} & A_{22}^{-1} B_{22}
\end{array}\right)\left(\begin{array}{c}
\Delta k_{t} \\
\Delta d_{t} \\
\Delta \xi_{t} \\
\Delta \zeta_{t}
\end{array}\right) \equiv J\left(\begin{array}{c}
\Delta k_{t} \\
\Delta d_{t} \\
\Delta \xi_{t} \\
\Delta \zeta_{t}
\end{array}\right)
$$

Tedious but straightforward computations give after simplifications the characteristic polynomial

$$
\begin{aligned}
\mathcal{P}(\lambda) & =\lambda^{4}-\lambda^{3}\left[\frac{\beta T_{k}^{* 2}}{b \epsilon_{c c} c^{*} T_{k k}^{*}}\left(\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{\beta+b^{2}}{\beta b}+\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right] \\
& +\lambda^{2}\left[\frac{(1+\beta) T_{k}^{* 2}}{b \epsilon_{c c} c^{*} T_{k k}^{* k}}\left(\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{\beta+b^{2}}{\beta b}\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{2}{\beta}\right] \\
& -\lambda\left[\frac{T_{k}^{* 2}}{b \epsilon_{c c} c^{*} T_{k k}^{* k}}\left(\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{1}{\beta}\left(\frac{\beta+b^{2}}{\beta b}+\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)\right]+\frac{1}{\beta^{2}}
\end{aligned}
$$

After simplifications we get the expression (12).

### 7.3 Proof of Lemma 2

Under Assumption 4, let us denote the two degree-2 polynomials as follows

$$
\begin{equation*}
\mathcal{P}_{1}(\lambda)=\lambda^{2}-\lambda\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{1}{\beta}, \quad \mathcal{P}_{2}(\lambda)=\frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \tag{34}
\end{equation*}
$$

The discriminant of $\mathcal{P}_{1}(\lambda)$ is equal to:

$$
\Delta_{1}=\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}+\frac{2}{\sqrt{\beta}}\right)\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}-\frac{2}{\sqrt{\beta}}\right)
$$

Using (5)-(6) we get

$$
\begin{aligned}
& \Delta_{1}=\left(\frac{1}{u_{c d}}\right)^{2}\left(u_{c c}+\frac{2 u_{c d}}{\sqrt{\beta}}+\frac{u_{d d}}{\beta}\right)\left(u_{c c}-\frac{2 u_{c d}}{\sqrt{\beta}}+\frac{u_{d d}}{\beta}\right) \\
& =\left(\frac{1}{u_{c d}}\right)^{2}\left(\begin{array}{cc}
1 & \frac{1}{\sqrt{\beta}}
\end{array}\right)\left(\begin{array}{ll}
u_{c c} & u_{c d} \\
u_{d c} & u_{d d}
\end{array}\right)\binom{1}{\frac{1}{\sqrt{\beta}}} \\
& \times\left(\begin{array}{ll}
1 & -\frac{1}{\sqrt{\beta}}
\end{array}\right)\left(\begin{array}{ll}
u_{c c} & u_{c d} \\
u_{d c} & u_{d d}
\end{array}\right)\binom{1}{-\frac{1}{\sqrt{\beta}}}
\end{aligned}
$$

Under the concavity property in Assumption 2, the Hessian matrix of the utility function $u(c, d)$ is quasi-negative definite which implies $\beta_{1} \geq 0$ and the associated characteristic roots are necessarily real. From $\mathcal{P}_{2}(\lambda)$ we obviously conclude that for any sign of the capital intensity difference $b$ the associated characteristic roots are also necessarily real.

### 7.4 Proof of Proposition 2

Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{c d}, \epsilon_{d c} \geq 0$, i.e. $u_{c d} \leq 0$. Using the fact that $\frac{\epsilon_{c c}}{\epsilon_{d c}}=\frac{\epsilon_{c d}}{\epsilon_{d d}}$, we derive the following expression

$$
\begin{equation*}
\mathcal{P}_{1}(\lambda)=\left(\lambda-\frac{\epsilon_{c c}}{\epsilon_{d c}}\right)\left(\lambda-\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}\right) \tag{35}
\end{equation*}
$$

The associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are therefore both positive. Moreover we get:

$$
\begin{aligned}
& \mathcal{P}_{1}(0)=\frac{1}{\beta} \geq 1 \\
& \mathcal{P}_{1}(1)=-\epsilon_{c c} \epsilon_{d c}\left(\frac{1}{\epsilon_{c c}}-\frac{1}{\epsilon_{d c}}\right)\left(\frac{1}{\beta \epsilon_{c c}}-\frac{1}{\epsilon_{d c}}\right)
\end{aligned}
$$

The normality Assumption 3 implies $\mathcal{P}_{1}(1)<0$ and we conclude that the associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are such that $\lambda_{1}<1$ and $\lambda_{2}>1$.

From $\mathcal{P}_{2}(\lambda)$, the associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are both positive. Moreover we derive:

$$
\mathcal{P}_{2}(0)=\frac{1}{\beta} \geq 1, \quad \mathcal{P}_{2}(1)=-\frac{(\beta-b)(1-b)}{\beta b}
$$

From constant returns to scale, we get $w a_{01}+r a_{11}=p$ with $a_{01}=l_{1} / y$ and $a_{11}=k_{1} / y$. The second equation in (11) rewrites as $p=\beta r$. We then obtain after substitution in the previous equation $r\left(\beta-a_{11}\right)=w a_{01}>0$ and thus

$$
\beta-b=\frac{a_{00}\left(\beta-a_{11}\right)+a_{10} a_{01}}{a_{00}}>0
$$

When $b \geq 0$ we then necessarily have $b<\beta \leq 1$. It follows that $\mathcal{P}_{2}(0)<0$ and we conclude that the associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are such that $\lambda_{1}<1$ and $\lambda_{2}>1$. The steady state is therefore a saddle-point.

### 7.5 Proof of Proposition 3

i) Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{c d}, \epsilon_{d c}<0$, i.e. $u_{c d}>0$. As shown previously, we derive from $\mathcal{P}_{2}(\lambda)=0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_{1}(\lambda)$ as given by (35), the associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are both negative. Moreover, we get:

$$
\mathcal{P}_{1}(-1)=\left(1+\frac{\epsilon_{c c}}{\epsilon_{d c}}\right)\left(1+\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}\right)=\frac{\left(\epsilon_{c c}+\epsilon_{d d}\right)\left(\beta \epsilon_{c c}+\epsilon_{d c}\right)}{\beta \epsilon_{c c} \epsilon_{d c}}
$$

We conclude easily that

$$
\begin{aligned}
& \mathcal{P}_{1}(-1)<0 \Leftrightarrow \epsilon_{c c} \in\left(0,-\epsilon_{d c}\right) \cup\left(-\epsilon_{d c} / \beta,+\infty\right) \\
& \mathcal{P}_{1}(-1)>0 \Leftrightarrow \epsilon_{c c} \in\left(-\epsilon_{d c},-\epsilon_{d c} / \beta\right)
\end{aligned}
$$

It follows that the steady state is a saddle-point with damped oscillations when $\epsilon_{c c} \in\left(0,-\epsilon_{d c}\right) \cup\left(-\epsilon_{d c} / \beta,+\infty\right)$ and there exists a flip bifurcation with persistent period-2 cycles when $\epsilon_{c c}$ crosses the bifurcation values $-\epsilon_{d c}$ or $-\epsilon_{d c} / \beta$.
ii) Under Assumptions $1-4$, let $\epsilon_{c d}, \epsilon_{d c} \geq 0$, i.e. $u_{c d} \leq 0$, and $b<0$. As shown previously, we derive from $\mathcal{P}_{1}(\lambda)=0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_{2}(\lambda)$, the associated characteristic roots $\lambda_{1}$ and $\lambda_{2}$ are both negative. Moreover we get:

$$
\mathcal{P}_{2}(-1)=\frac{(1+b)(b+\beta)}{\beta b}
$$

We conclude easily that

$$
\begin{aligned}
& \mathcal{P}_{1}(-1)<0 \Leftrightarrow b \in(-\infty,-1) \cup(-\beta, 0) \\
& \mathcal{P}_{1}(-1)>0 \Leftrightarrow b \in(-1,-\beta)
\end{aligned}
$$

It follows that the steady state is a saddle-point with damped oscillations when $b \in(-\infty,-1) \cup(-\beta, 0)$. Moreover, if there is some $\beta^{*} \in(0,1)$ such that $b \in\left(-\infty,-\beta^{*}\right)$, then there exists $\bar{\beta} \in(0,1)$ such that, when $\beta$ crosses $\bar{\beta}$ from above, $\left(k^{*}, d^{*}\right)$ undergoes a flip bifurcation leading to persistent period2 cycles.
iii) The case where the consumption good is capital intensive, i.e. $b<0$, and $\epsilon_{c d}, \epsilon_{d c}<0$, i.e. $u_{c d}>0$, is obviously derived from the two previous cases.

### 7.6 Proof of Corollary 1

Under a linear homogeneous utility function, standard Euler equalities based on the homogeneity of degree 1 , namely $u=u_{c} c+u_{d} B d, 0=u_{c c} c+u_{c d} B d$ and $0=u_{d c} c+u_{d d} B d$, lead to

$$
u_{c d}=-\frac{u_{c c}}{B d}, \quad u_{d c}=-\frac{u_{d d} B d}{c} \quad \text { and thus } u_{d d}=u_{c c}\left(\frac{c}{B d}\right)^{2}
$$

Moreover, we get from the first order condition $u_{d} B=\beta u_{c}$ and (13)

$$
\frac{c}{B d}=\frac{\beta \phi}{1-\phi}
$$

Substituting all this into (5)-(6) implies

$$
\epsilon_{c d}=-\epsilon_{c c}, \quad \epsilon_{d c}=-\epsilon_{c c} \frac{1-\phi}{\phi}, \quad \epsilon_{d d}=\epsilon_{c c} \frac{1-\phi}{\phi}
$$

The result follows from Proposition 3.

### 7.7 Proof of Proposition 4

The characteristic polynomial (12) can be expressed as follows
$\left[\lambda^{2}-\lambda\left(\frac{\epsilon_{d c}}{\beta \epsilon_{c c}}+\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)+\frac{1}{\beta}\right] \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b}=\lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\beta T_{k}^{* 2}}{b \epsilon_{c c} c^{*} T_{k k}^{*}}\left(\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}\right)$
or equivalently, using the notations of Lemma 2,

$$
P_{1}(\lambda) P_{2}(\lambda)=P_{3}(\lambda)
$$

with $P_{3}(\lambda)$ a degree- 3 polynomial while $P_{1}(\lambda) P_{2}(\lambda)$ is a degree- 4 polynomial. If these two polynomials intersect four times, then the four characteristic roots are real. To determine the number of intersections of these polynomials, we can use informations derived from the location of their respective roots. The roots of $P_{3}(\lambda)=0$ are quite obvious, namely $\lambda_{31}=0, \lambda_{32}=1$ and $\lambda_{33}=1 / \beta$. Moreover, depending of the sign of $\epsilon_{c d}, \epsilon_{d c}$ we get

- if $\epsilon_{c d}, \epsilon_{d c}<0$, then $\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}>$ and $\lim _{\lambda \rightarrow+\infty} P_{3}(\lambda)=-\infty$ while $\lim _{\lambda \rightarrow \infty} P_{3}(\lambda)=+\infty$;
- if $\epsilon_{c d}, \epsilon_{d c}>0$, then $\frac{\epsilon_{c c}}{\epsilon_{d c}}-\frac{\epsilon_{c d}}{\epsilon_{d d}}<$ and $\lim _{\lambda \rightarrow+\infty} P_{3}(\lambda)=+\infty$ while $\lim _{\lambda \rightarrow \infty} P_{3}(\lambda)=-\infty$;

The roots of $P_{1}(\lambda) P_{2}(\lambda)=0$ are obviously given by the respective roots of $P_{1}(\lambda)=0$ and $P_{2}(\lambda)=0$.
i) Assume first that $b>0$. We have shown in the proof of Proposition 2 that $b<\beta \leq 1$. The roots of $P_{2}(\lambda)=0$ are then quite obvious, namely $\lambda_{21}=$ $1 / b>1$ and $\lambda_{22}=b / \beta<1$. Finally, the roots of $P_{1}(\lambda)=0$ are necessarily real and negative if $\epsilon_{c d}, \epsilon_{d c}<0$, or positive if $\epsilon_{c d}, \epsilon_{d c}>0$. Moreover, we have $\lim _{\lambda \rightarrow \pm \infty} P_{1}(\lambda) P_{2}(\lambda)=+\infty$ and $P_{1}(0) P_{2}(0)>0$.

If $\epsilon_{c d}, \epsilon_{d c}<0$, we derive from the above informations that $P_{1}(b / \beta) P_{2}(b / \beta)=0>P_{3}(b / \beta)$ while $P_{1}(1) P_{2}\left(1<P_{3}(b / \beta)=0\right.$ implying a first intersection between $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$ in the positive orthant. Moreover, since $P_{1}(1 / \beta) P_{2}(1 / \beta)<P_{3}(1 / \beta)=0$ while $P_{1}(1 / b) P_{2}(1 / b)=$ $0>P_{3}(b / \beta)$, we get a second intersection $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$ in the positive orthant. Since $P_{1}(0) P_{2}(0)>0, P_{1}(\lambda) P_{2}(\lambda)=0$ admits two roots in the negative horthant, $P_{3}(0)=0$ and $P_{3}(\lambda)$ is an increasing function in the negative hortant, we conclude that there necessarily exists a third intersection between $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$ in the positive orthant. The last intersection, which also occurs in the negative orthant, is obtained because $\lim _{\lambda \rightarrow-\infty} P_{1}(\lambda) P_{2}(\lambda)>\lim _{\lambda \rightarrow-\infty} P_{3}(\lambda)$. Indeed $P_{3}(\lambda)$ a degree- 3 polynomial while $P_{1}(\lambda) P_{2}(\lambda)$ is a degree- 4 polynomial. We then get the following graphical illustration


It follows that the four roots of the characteristic polynomial (12) are
real.
If $\epsilon_{c d}, \epsilon_{d c}>0$, the roots of $P_{3}(\lambda)=0$ and $P_{2}(\lambda)=0$ are the same as before while the roots of $P_{1}(\lambda)=0$ are now real and positive. Since $P_{1}(0) P_{2}(0)>0, P_{1}(1 / b) P_{2}(1 / b)=0$ and $P_{1}(1) P_{2}(1)>0$, there necessarily exists a second root of $P_{1}(\lambda) P_{2}(\lambda)=0$ between 0 and $1 / b$ implying two intersections between $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$. The two others are obtained since $P_{1}(1 / \beta) P_{2}(1 / \beta)>P_{3}(1 / \beta)=0, P_{1}(b / \beta) P_{2}(b / \beta)=0<P_{3}(b / \beta)$ and $\lim _{\lambda \rightarrow+\infty} P_{1}(\lambda) P_{2}(\lambda)>\lim _{\lambda \rightarrow+\infty} P_{3}(\lambda)$. We then get the following graphical illustration


Here again, it follows that the four roots of the characteristic polynomial (12) are real.
ii) Assume now that $b<0$ and $\epsilon_{c d}, \epsilon_{d c}>0$. The roots of $P_{2}(\lambda)=0$ become negative, namely $\lambda_{21}=1 / b<\lambda_{22}=b / \beta<0$. We easily get $P_{1}(0) P_{2}(0)>0, P_{1}(1) P_{2}(1)<P_{3}(1)=0, P_{1}(1 / \beta) P_{2}(1 / \beta)<P_{3}(1 / \beta)=0$, $\lim _{\lambda \rightarrow+\infty} P_{1}(\lambda) P_{2}(\lambda)=+\infty$ and $\lim _{\lambda \rightarrow+\infty} P_{3}(\lambda)=-\infty$. It follows that there are three intersections between $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$ in the positive orthant. Moreover, we have $\lim _{\lambda \rightarrow-\infty} P_{1}(\lambda) P_{2}(\lambda)>\lim _{\lambda \rightarrow-\infty} P_{3}(\lambda)$ implying the existence of two additional intersections between $P_{1}(\lambda) P_{2}(\lambda)$ and $P_{3}(\lambda)$ in the negative orthant. We then get the following graphical illustration
and it follows that the four roots of the characteristic polynomial (12) are real.


### 7.8 The Cobb-Douglas example

We follow the same methodology as in Baierl et al. [2]. Consider the CobbDouglas production functions as given by (25). The Lagrangian associated with the optimization program (1) is:
$\mathcal{L}=k_{0}^{\alpha_{0}} l_{0}^{1-\alpha_{0}}+w\left(1-l_{0}-l_{1}\right)+r\left(k-k_{0}-k_{1}\right)+p\left[k_{1}^{\alpha_{1}} l_{1}^{1-\alpha_{1}}-y\right]$
The first order conditions are:

$$
\begin{align*}
r & =\alpha_{0} k_{0}^{\alpha_{0}-1} l_{0}^{1-\alpha_{0}}=p \alpha_{1} k_{1}^{\alpha_{1}-1} l_{1}^{1-\alpha_{1}}  \tag{37}\\
w & =\left(1-\alpha_{0}\right) k_{0}^{\alpha_{0}} l_{0}^{-\alpha_{0}}=p\left(1-\alpha_{1}\right) k_{1}^{\alpha_{1}} l_{1}^{-\alpha_{1}} \tag{38}
\end{align*}
$$

Using $k_{0}=k-k_{1}, l_{0}=1-l_{1}$, and merging the above equations gives:

$$
\begin{align*}
l_{0}^{*} & =\frac{\left(1-\alpha_{0}\right) \alpha_{1}\left(k-k_{1}^{*}\right)}{\left(\alpha_{0}-\alpha_{1}\right) k_{1}^{*}+\left(1-\alpha_{0}\right) \alpha_{1} k}  \tag{39}\\
l_{1}^{*} & =\frac{\alpha_{0}\left(1-\alpha_{1}\right) k_{1}^{*}}{\left(\alpha_{0}-\alpha_{1}\right) k_{1}^{*}+\left(1-\alpha_{0}\right) \alpha_{1} k}  \tag{40}\\
K_{c}^{*} & =k-k_{1}^{*}  \tag{41}\\
k_{1}^{*} & =g(k, y) \equiv g \tag{42}
\end{align*}
$$

where

$$
\begin{equation*}
g(k, y)=\left\{k_{1} \in\left[0, k^{\alpha_{1}}\right] / y=\frac{\left[\alpha_{0}\left(1-\alpha_{1}\right)\right]^{1-\alpha_{1}} k_{1}}{\left[\left(1-\alpha_{0}\right) \alpha_{1} k+\left(\alpha_{0}-\alpha_{1}\right) k_{1}\right]^{1-\alpha_{1}}}\right\} \tag{43}
\end{equation*}
$$

From the envelope theorem we get:

$$
T_{k}=r^{*}, \quad T_{y}=-p^{*}
$$

From (37), (39) and (41) we obtain:

$$
\begin{equation*}
r^{*}=\alpha_{0}\left[\frac{\left(1-\alpha_{0}\right) \alpha_{1}}{\left(1-\alpha_{0}\right) \alpha_{1} k+\left(\alpha_{0}-\alpha_{1}\right) g}\right]^{1-\alpha_{0}} \tag{44}
\end{equation*}
$$

and from (37), (40), (42) and (44):

$$
\begin{equation*}
p^{*}=\frac{\alpha_{0}\left[\left(1-\alpha_{0}\right) \alpha_{1}\right]^{1-\alpha_{0}}\left[\alpha_{0}\left(1-\alpha_{1}\right)\right]^{-\left(1-\alpha_{1}\right)}\left[\left(1-\alpha_{0}\right) \alpha_{1} k+\left(\alpha_{0}-\alpha_{1}\right) g\right]^{\alpha_{0}-\alpha_{1}}}{\alpha_{1}} \tag{45}
\end{equation*}
$$

By the derivation of $g$, we have, for any equilibrium path, the identity ( $1-$ $\left.\alpha_{0}\right) \alpha_{1} k+\left(\alpha_{0}-\alpha_{1}\right) g=\alpha_{0}\left(1-\alpha_{1}\right)(g / y)^{1 /\left(1-\alpha_{1}\right)}$. Substituting this into (44) and (45) gives after simplifications:

$$
\begin{aligned}
T_{k}(k, y) & =\alpha_{0}\left(\frac{\left(1-\alpha_{0}\right) \alpha_{1}}{\alpha_{0}\left(1-\alpha_{1}\right)}\right)^{1-\alpha_{0}}\left(\frac{y}{g}\right)^{\frac{1-\alpha_{0}}{1-\alpha_{1}}} \\
T_{y}(k, y) & =-\frac{\alpha_{1}}{\beta_{1}}\left(\frac{\left(1-\alpha_{0}\right) \alpha_{1}}{\alpha_{0}\left(1-\alpha_{1}\right)}\right)^{1-\alpha_{0}}\left(\frac{y}{g}\right)^{\frac{\alpha_{1}-\alpha_{0}}{1-\alpha_{1}}} \\
T_{k k}(k, y) & =-T_{k}(k, y) \frac{g_{1}}{g}
\end{aligned}
$$

with $g_{1}=\partial g(k, y) / \partial k$. A steady state $k^{*}$ is then defined as $T_{k}\left(k^{*}, k^{*}\right)+$ $\beta T_{y}\left(k^{*}, k^{*}\right)$. Denote $g^{*}=g\left(k^{*}, k^{*}\right)$ and $y^{*}=k^{*}$. Using the derivatives of $T$ in the definition of $k^{*}$ gives:

$$
\begin{equation*}
g^{*}=\beta \alpha_{1} k^{*} \tag{46}
\end{equation*}
$$

Substituting (46) into the definition of $g$, we find

$$
\begin{equation*}
k^{*}=\frac{\alpha_{0}\left(1-\alpha_{1}\right)\left(\beta \alpha_{1}\right)^{\frac{1}{1-\alpha_{1}}}}{\alpha_{1}\left[1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)\right]} \tag{47}
\end{equation*}
$$

Considering (43), we easily derive

$$
\begin{equation*}
g_{1}=\frac{\beta \alpha_{1}\left(1-\alpha_{0}\right)\left(1-\alpha_{1}\right)}{1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)} \tag{48}
\end{equation*}
$$

From all these results and (4), we get

$$
\begin{aligned}
c^{*}=T\left(k^{*}, k^{*}\right) & =\left(\frac{\alpha_{0}\left(1-\alpha_{1}\right)}{\left(1-\alpha_{0}\right) \alpha_{1}}\right)^{\alpha_{1}} \frac{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\beta \alpha_{1}\right)^{\frac{\alpha_{0}}{1-\alpha_{1}}}}{1-\alpha_{0}+\beta\left(\alpha_{0}-\alpha_{1}\right)} \\
r^{*}=T_{k}\left(k^{*}, k^{*}\right) & =\alpha_{0}\left(\frac{\left(1-\alpha_{0}\right) \alpha_{1}}{\alpha_{0}\left(1-\alpha_{1}\right)}\right)^{1-\alpha_{0}}\left(\beta \alpha_{1}\right)^{-\frac{1-\alpha_{0}}{1-\alpha_{1}}} \\
T_{k k}\left(k^{*}, k^{*}\right) & =-\frac{T_{k}\left(k^{*}, k^{*}\right)}{k^{*}} \frac{\left(1-\alpha_{0}\right)^{2}}{1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)} \\
b & =\frac{\beta\left(\alpha_{1}-\alpha_{0}\right)}{1-\alpha_{0}}
\end{aligned}
$$

### 7.9 Proof of Proposition 5

The characteristic polynomial (27) can be expressed as $\mathcal{Q}_{1}(\lambda)=\mathcal{Q}_{2}(\lambda)$ with
$\mathcal{Q}_{1}(\lambda) \equiv \frac{1}{\gamma-\phi}\left[\lambda^{2}(\gamma-\phi)+\lambda\left(\frac{(\gamma-\phi)^{2}+\beta \phi^{2}-\beta \phi \epsilon_{c c}(1-\gamma)(2 \phi-\gamma)}{\beta \phi\left[1-\epsilon_{c c}(1-\gamma]\right.}\right)+\frac{(\gamma-\phi)}{\beta}\right] \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b}$
$\mathcal{Q}_{2}(\lambda) \equiv \frac{1}{\gamma-\phi} \lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]}{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)} \frac{\epsilon_{c c}(1-\gamma)\left[\gamma-\epsilon_{c c} \phi(1-\gamma)\right]}{\left[1-\epsilon_{c c}(1-\gamma)\right]}$
Considering the limit $\phi \rightarrow \gamma$ we immediately conclude that one root $\lambda_{1}$ is necessarily real and equal to $\pm \infty$ and we get

$$
\begin{aligned}
& \mathcal{Q}_{1}(\lambda)=\lambda \gamma \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \\
& \mathcal{Q}_{2}(\lambda)=\lambda \gamma(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]}{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)} \epsilon_{c c}(1-\gamma)
\end{aligned}
$$

It follows that a second root $\lambda_{2}$ is real and equal to 0 . Computing now the derivatives $\mathcal{Q}_{1}^{\prime}(\lambda)$ and $\mathcal{Q}_{2}^{\prime}(\lambda)$, and evaluating them at 0 gives

$$
\begin{aligned}
& \mathcal{Q}_{1}^{\prime}(0)=\frac{1}{\beta} \\
& \mathcal{Q}_{2}^{\prime}(0)=\frac{1}{\beta} \frac{\alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]}{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)} \epsilon_{c c}(1-\gamma)
\end{aligned}
$$

It follows that $\mathcal{Q}_{1}^{\prime}(0) \gtrless \mathcal{Q}_{2}^{\prime}(0)$ if and only if $\epsilon_{c c} \lessgtr \hat{\epsilon}_{c c}$ with

$$
\hat{\epsilon}_{c c} \equiv \frac{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)}{(1-\gamma) \alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]} \in\left(0, \tilde{\epsilon}_{c c}\right)
$$

We conclude therefore that there exist two additional intersections between $\mathcal{Q}_{1}(\lambda)$ and $\mathcal{Q}_{2}(\lambda)$ implying that the two last characteristic roots $\lambda_{3}, \lambda_{4}$ are also real. Let us then assume that $b \in(-\infty,-1) \cup(-\beta, 0)$. We derive that
i) if $\epsilon_{c c}<\hat{\epsilon}_{c c}$ then $\mathcal{Q}_{1}^{\prime}(0)>\mathcal{Q}_{2}^{\prime}(0)$ with $\mathcal{Q}_{1}(1 / b)=\mathcal{Q}_{1}(b / \beta)=0$ which implies that one intersection must occur between -1 and 0 , say $\lambda_{3} \in(-1,0)$. Moreover we derive also that $\lambda_{1}=-\infty$ and $\lambda_{4}<-1$;
ii) if $\epsilon_{c c} \in\left(\hat{\epsilon}_{c c}, \tilde{\epsilon}_{c c}\right)$ then $\mathcal{Q}_{1}^{\prime}(0)<\mathcal{Q}_{2}^{\prime}(0)$ with $\mathcal{Q}_{2}(1)=0$ which implies that one intersection must occur between 0 and 1 , say $\lambda_{3} \in(0,1)$. Moreover we derive $\lambda_{1}=+\infty$ and $\lambda_{4}>1$.
We then conclude by continuity that there exists $0<\bar{\phi}<\gamma$ such that when $\phi \in(\bar{\phi}, \gamma)$, the above results hold with $\lambda_{1} \in(-\infty,-1)$ and $\lambda_{2} \in(-1,0)$ when $\epsilon_{c c}<\hat{\epsilon}_{c c}$ or $\lambda_{1} \in(1, \infty)$ and $\lambda_{2} \in(0,1)$ when $\epsilon_{c c} \in\left(\hat{\epsilon}_{c c}, \tilde{\epsilon}_{c c}\right)$.

Note now that the characteristic polynomial (27) can be also expressed as $\mathcal{Q}_{1}(\lambda)=\mathcal{Q}_{2}(\lambda)$ with

$$
\begin{aligned}
& \mathcal{Q}_{1}(\lambda) \equiv \frac{1}{\phi}\left[\lambda^{2} \phi+\lambda\left(\frac{(\gamma-\phi)^{2}+\beta \phi^{2}-\beta(\gamma-\phi) \epsilon_{c c}(1-\gamma)(2 \phi-\gamma)}{\beta \phi\left[1-\epsilon_{c c}(1-\gamma]\right.}\right)+\frac{\phi}{\beta}\right] \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \\
& \mathcal{Q}_{2}(\lambda) \equiv \frac{1}{\phi} \lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right) \frac{\alpha_{0}\left[1-\alpha_{0}+\beta \alpha_{1}\left(\alpha_{0}-\alpha_{1}\right)\right]}{\left(1-\alpha_{0}\right)\left(1-\beta \alpha_{1}\right)\left(\alpha_{0}-\alpha_{1}\right)} \frac{\epsilon_{c c} \phi(1-\gamma)\left[\gamma-\epsilon_{c c} \phi(1-\gamma)\right]}{(\gamma-\phi)\left[1-\epsilon_{c c}(1-\gamma)\right]}
\end{aligned}
$$

Considering the limit $\phi \rightarrow 0$ we immediately conclude that one root $\lambda_{1}$ is necessarily real and equal to $-\infty$ as $b<0$, and we get

$$
\begin{aligned}
& \mathcal{Q}_{1}(\lambda)=\frac{\lambda \gamma^{2}}{\beta\left[1-\epsilon_{c c}(1-\gamma)\right]} \frac{(\lambda b-1)(\lambda \beta-b)}{\beta b} \\
& \mathcal{Q}_{2}(\lambda)=0
\end{aligned}
$$

It follows that $\lambda_{2}=0, \lambda_{3}=1 / b$ and $\lambda_{4}=b / \beta$ with one larger than -1 and the other lower than -1 as $b \in(-\infty,-1) \cup(-\beta, 0)$. We then conclude by continuity that there exists $0<\phi<\bar{\phi}$ such that when $\phi \in(0, \underline{\phi})$, the above results hold with $\lambda_{1} \in(-\infty,-1)$ and $\lambda_{2} \in(-1,0)$.

### 7.10 Proof of Proposition 7

As shown in the proof of Proposition 1, there exists a unique steady state $\left(k^{*}, d^{*}\right)$ solution of equations $R^{*}=r^{*} / p^{*}=\beta^{-1}$ and $u_{d}\left(c^{*}, B d^{*}\right)=$ $\beta u_{c}\left(c^{*}, B d^{*}\right)$. Moreover, $k^{*}$ does not depend on the utility function $u(c, B d)$. Since the stationary bequest $x^{*}$ is strictly positive if and only if $r^{*} k^{*}=T_{k}\left(k^{*}, k^{*}\right) k^{*}>d^{*}$, let us consider a particular value $d^{*}=\bar{d} \in$ $\left(0, \min \left\{T_{k}\left(k^{*}, k^{*}\right), T_{k}\left(k^{*}, k^{*}\right) k^{*}\right\}\right)$. Then, for any $\beta \in(0,1)$, the same argument as in the proof of Proposition 1 holds: there generically exists a unique value $B^{*}$ such that when $B=B^{*}, d^{*}=\bar{d}$ is a normalized steady state such that $x^{*}>0$.

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[^1]:    ${ }^{1}$ An increasing population could be considered without altering all our results.
    ${ }^{2}$ Considering that in an OLG model one period is approximately 30 years, complete depreciation is a realistic assumption.

[^2]:    ${ }^{3}$ In the case $\beta=1$, the infinite sum into the optimization program (8) may not converge. In such a case we may apply the definition of optimality as provided by Ramsey [15].

[^3]:    ${ }^{4}$ These results are derived from concavity and standard Euler equalities for homogeneous functions, namely $u(c, B d)=u_{c}(c, B d) c+u_{d}(c, B d) B d, 0=u_{c c}(c, B d) c+$ $u_{c d}(c, B d) B d$ and $0=u_{d c}(c, B d) c+u_{d d}(c, B d) B d$.
    ${ }^{5}$ We do not need to introduce a normalization constant $B$ in such a simple example.

[^4]:    ${ }^{6}$ Standard Euler equalities for homogeneous functions become now $\gamma u(c, B d)=$ $u_{c}(c, B d) c+u_{d}(c, B d) B d,(\gamma-1) u_{c}(c, B d)=u_{c c}(c, B d) c+u_{c d}(c, B d) B d$ and $(\gamma-$ 1) $u_{d}(c, B d)=u_{d c}(c, B d) c+u_{d d}(c, B d) B d$.

[^5]:    ${ }^{7}$ See Benhabib and Nishimura [6], Cartigny and Venditti [9], Venditti [17].

