Is the preference of the majority representative ?

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Abstract

Given a profile of preferences on a set of alternatives, a majority relation is a complete binary relation that aggrees with the strict preference of a strict majority of these preferences whenever such strict strict majority is observed. We show that a majority binary relation is, among all conceivable binary relations, the most representative of the profile of preferences from which it emanates. We define "the most representative" to mean "the closest in the aggregate". This requires a definition of what it means for a pair of preferences to be closer to each other then another. We assume that this definition takes the form of a distance function defined over the set of all conceivable preferences. We identify a necessary and sufficient condition for such a distance to be minimized by the preference of the majority. This condition requires the distance to be additive with respect to a plausible notion of compromise between preferences. The well-known Kemeny distance between preference does satisfy this property. We also provide a characterization of these class of distances as numerical representation of a primitive qualitative proximity relation between preferences.

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1 Introduction

The "preference of the majority" is indisputably one of the most widely used and discussed social preference. Yet, the normative justifications in favour of the "majoritarian" way of aggregating individual preferences are surprisingly thin. An important such justification has been provided by May

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(1952), who proves that when there are only two alternatives, the majority rule is the only mapping of individual preferences into social ranking that is *decisive*, *egalitarian*, *neutral* and *positively responsive*. A well-known limitation of the majority rule, at least since Condorcet in the late XVIIIth century, is its failure to satisfy transitivity. This limitation is obviously not addressed by May (1952) who considers only the case where two alternatives are concerned. In the discussion of his famous impossibility theorem, Arrow (1963) himself (see e.g. p. 101), recognizes that the generalization to more than two alternatives of May's results was not easy. Papers who have done so include Dasgupta and Maskin (2008) and the recent work by Horan, Osborne, and Sanver (2018).

In this paper, we propose an alternative justification for the Majority rule than that developed in the May (1952) tradition. Specifically, we show that the preference of the majority qualifies, in a somewhat strong sense, as being representative of the collection of preferences from which it emanates. The notion of representativeness on which our argument is constructed is that underlying the choices of several measures of "central tendency" in classical statistics. A common justification indeed for the mean of a set of numbers as a "representative statics" for these numbers is that the mean minimizes the sum of the squares of the differences itself and the represented numbers. Similarly, the median of a set of numbers - another widely used measure of "central tendency" - is commonly justified by the fact that it minimizes the sum of the absolute values of those same difference, while the mode minimizes a somewhat more degenerate distances between numbers that is 1 if the numbers differ and 0 if they don't. Similarly, it is common in regression analysis to fit a cloud of points indicating values taken by a dependant variable and a collection of "independent" ones by a specific function whose parameters are "estimated" by minimizing the sum of the (square of) the discrepancies between the predicted and observed values of the dependant variables. The parametric curve estimated in this fashion is commonly portrayed as "representative" of the cloud of points

In this paper, we show that the "preference of the majority" represents in a similar fashion the individual preferences in the sense that it minimizes the sum of distances between itself and the preferences for some distance function that represents an underlying notion of preference dissimilarity. What such a notion can be is, of course, far from clear. Because of this, we take the prudent view of not specifying too much the notion of preference dissimilarity. We actually identify the properties of the family of preference dissimilarity notions that are necessary and sufficient for the distance that numerically represent them to be minimized by a majoritarian preference. This family is that of all distances that are *additive* with respect to any two of three preferences that are connected by a notion of preference "compromise" The notion of compromise considered is that underlying the Pareto principle. That is to say, a compromise between two preferences is any preference that agrees with the unanimity of the two preferences when this unanimity is observed.

Our analysis can be seen as a generalization of a small literature in social choice that has discussed the representativeness of the majoritarian preference in the sense of distance minimization with respect to the specific *Kemeny* notion of proximity (see e.g. Kemeny (1959) and Kemeny and Snell (1962)). It has been known indeed for quite a while that, when transitive, the majority relation maximizes the sum of pairwise agreements between itself and the individual preferences (see e.g. Monjardet (2005)). In other words, when the majority relation of a vote profile is transitive, it is the unique Kemeny distance-minimizing social welfare relation (Demange (2012)). This literature has also established that the majority rule can then be seen as the 'median' preference in a metric space over preferences in which the metric is the Kemeny distance. For example, Young and Levenglick (1978) have characterized in this fashion all Condorcet consistent rules. Other contributions to this literature include Lerer and Nitzan (1985) and Andjiga, Mekuko, and Moyouwou (2014). There are also some work like Bossert and Storcken (1992) and Nehring and Puppe (2007) that studies the existence of 'suitable' strategy-proof social welfare functions in such median spaces. In this paper, we therefore extend the results about the "representativeness" of the majority by showing that it holds for a larger class of notions of proximity than that of Kemeny that we precisely identify, through the property of between-additivity. We also shows that the representativeness of majoritarian preferences holds also in the (very frequent) case where those majoritarian preferences are not transitive.

The plan of the remaining of the paper is as follows. In the next section, we introduce the notation and the model. Section 2 states and prove the results and section 3 concludes

2 The Model

2.1 Notation

We are interested in problems involving variable collections of preferences over a finite set X of at least 3 alternatives. Since preferences are described as binary relations, we first introduce our notation pertaining to those. By a *binary relation* R on X, we mean a subset of X^2. Given a binary relation R on X, we define its *symmetric* factor R_S by $(x,y) \in R_S \iff \{(x,y),(y,x)\} \subset R$ and its *asymmetric* factor R_A by $(x,y) \in R_A \iff [(x,y) \in R \text{ and } (y,x) \notin R]$. A binary relation R is *asymmetric* when it coincides with its asymmetric factor. A binary relation R on X is:

(i) *reflexive* if $(x,x) \in R$ for every $x \in X$.

(ii) *linear* if for no distinct x and y does the statement $(x,y) \in R_S$ hold.

(iii) *transitive* if, for any x, y and $z \in X$, $(x,z) \in R$ always follows $(x,y) \in R$ and $(y,z) \in R$

(iv) *complete* if $\{(x,y),(y,x)\} \cap R \neq \varnothing$ for every distinct $x, y \in X$.

A reflexive and transitive binary relation is called an *quasi-ordering*, and a complete quasi-ordering is called an *ordering*. We denote respectively by $\mathcal{C}$ and $\mathcal{R}$ the set of all complete binary relations and orderings on X. For any ordering R, we denote by X/R the quotient of X over R defined by: $X/R = \{A \subset X : \{(a,a'),(a',a)\} \subset R$ for any a and $a' \in A\}$. Hence the set X/R is the set of all classes of elements of X whose members are considered pairwise equivalent by R. It is well-know (and easy to check) that X/R is a partition of X if R is an ordering. Finaly, for any two binary relations R and R", we denote by $R \vartriangle R$" their symmetric set difference defined by $R \vartriangle R$" $= (R \cup R")\backslash(R \cap R")$.

We start the analysis by discussing a bit the notion of a compromise between individual preferences, that will always be depicted as complete and reflexive binary relations. After all, most efforts in social choice theory have been toward finding a plausible notion of such compromise. The cornerstone of the compromise's idea is that of a (Pareto) respect for unanimity. It seems indeed that any plausible notion of a compromise between two different preferences should respect the unanimity of those preferences whenever it occurs. This idea underlies the following notion of intermediateness, or betweenness, between two preferences. For any binary relation $R \subset X \times X$, we denote by $\widehat{R}$ its (possibly empty) non-trivial component defined by $\widehat{R} = R \cap [(X \times X)\backslash\{(x,y) \in X \times X : x = y\}]$. Hence, $\widehat{R}$ is the set of all pairs of distinct elements of X that are compared in one way or another by R. In what follows, we will often find useful to describe reflexive binary relations R by their non-trivial component $\widehat{R}$.

Definition 1 *For any two binary relations R and R" in $\mathcal{C}$, we say that the binary relation R' is between R and R" if only if* $(R \cap R") \subseteq R' \subseteq (R \cup R")$.

In words, R' is between R and R'' if R' always agrees with the unanimity of R and R" - when the latter occurs - and, somewhat conversely, never expresses a preference for one alternative over the other if this preference is not also expressed by either R or R". We observe trivially that this notion of betweenness is symmetric: R' is indeed between R and R" if and only if it is between R" and R. The definition of betweenness applies therefore to any three binary relations and generates as such a *ternary* relation on X. It turns out that an alternative - but actually equivalent - definition of betweenness can be formulated for *complete* binary relations. This equivalent definition makes, in our view, the notion of betweenness underlying Definition 1 even more intuitive.

Lemma 1 *Let R, R' and R'' be three complete binary relations on X. Then R' is between R and R'' as per Definition 1 if only if it satisfies:*

(i) $(x,y) \in R$ *and* $(x,y) \in R'' \Longrightarrow (x,y) \in R'$ *and,*
(ii) $(x,y) \in R_A$ *and* $(x,y) \in R''_A \Longrightarrow (x,y) \in R'_A$

Proof. *For one direction of the implication (that does not actually require completeness), let R,* R' *and* R'' *be three complete binary relations on* X *such that* R' *is between R and R" as per Definition 1. Since* $(R \cap R") \subseteq R'$*, Condition (i) of the Lemma follows. Assume now that* x *and* y *are two alternatives such that* $(x,y) \in R_A$ *and* $(x,y) \in R''_A$*. From the definition of the asymmetric factor of a binary relation, one has* $(x,y) \in R$ *and* $(x,y) \in R''$ *and, since* $(R \cap R") \subseteq R'$*, one must have* $(x,y) \in R'$*. We now show that* $(y,x) \notin R'$*. Suppose to the contrary that* $(y,x) \in R'$*. Since* $R' \subseteq (R \cup R'')$*, one must have either* $(y,x) \in R$ *or* $(y,x) \in R''$*. But neither of these statements is consistent with the fact that both* $(x,y) \in R_A$ *and* $(x,y) \in R''_A$ *hold.*
For the other direction of the implication, assume that R, R' *and* R'' *are three complete binary relations on* X *for which Statements (i) and (ii) of the lemma holds. Statement (i) clearly implies that* $(R \cap R") \subseteq R'$*. Consider now any two alternatives* x *and* y *in* X *such that neither* $(x,y) \in R$ *nor* $(x,y) \in R''$ *is true. We wish to show that* $(x,y) \in R'$ *does not hold. To see this, we observe that, since R and R" are complete, the fact that neither* $(x,y) \in R$ *nor* $(x,y) \in R''$ *is true implies that* $(y,x) \in R_A$ *and* $(y,x) \in R''_A$*. By Statement (ii) of the lemma, this implies that* $(y,x) \in R'_A$*, which implies in turn, from the very definition of the asymmetric factor of a binary relation, that* $(x,y) \notin R'$*, as required.* ■

Lemma 1 thus provides additional intuition about what it means for a preference to be "between" two others. A preference is between two others if and only it results from a (Paretian) compromise between those preferences. For any two preferences R and $R"$, we let $\mathcal{B}(R,R") = \{R' \in$R: $|R'$ is between R and $R"\}$. Since, for any two preferences R and $R"$, both R and $R"$ are (trivially) between R and $R"$, the set $\mathcal{B}(R,R")$ is never empty. Indeed, the notion of betweenness introduced by Definition 1 is a weak one that does not rule out the possibility that some (or all) of the three preferences R, R' and R'' involved in the definition be the same. This suggests the possibility of introducing the additional notion of *strict betweenness* as follows.

Definition 2 *For any two distinct binary relations R and R" in R, we say that* R' *is strictly between R and R" if only if one has* $(R \cap R") \subseteq R' \subseteq (R \cup R")$*,* $R' \neq R$ *and* $R" \neq R'$ *.*

For any two distinct relation R and $R"$ in $\mathcal{C}$, we let $\overline{\mathcal{B}(R,R")} = \{R' \in$R: $|R'$ is strictly between R and $R"\}$. The notion of strict betweenness just introduced opens the possibility for two distinct binary relations R and $R"$ in $\mathcal{C}$ to have no preference that lie strictly between them (and thus to have

$\overline{\mathcal{B}(R, R')} = \varnothing$). For example, if $X = \{a, b, c\}$, the distinct orderings R and R" defined by:

$$\begin{aligned} \widehat{R} &= \{(a,b),(b,c),(a,c)\} \text{ and,} \\ \widehat{R}'' &= \{(a,b),(b,a),(b,c),(a,c)\} \end{aligned}$$

have no ordering that lie strictly between them.

Any two distinct binary relations that have no preference that lie strictly between them will be called "adjacent". We formally defined this notion of adjacency between binary relations as follows.

Definition 3 *Two distinct binary relations R and R" in $\mathcal{C}$ are said to be adjacent if they are such that $\overline{\mathcal{B}(R, R')} = \varnothing$.*

The following lemma establishes an alternative definition of adjacency between two distinct preferences.

Lemma 2 *Two distinct binary relations R and R" in $\mathcal{C}$ are adjacent as per Definition 3 if and only if they are such that $\#(R \vartriangle R") = 1$.*

Proof. *Suppose first that R and R" are two distinct binary relation such that $\#(R \vartriangle R") = 1$. Since R and R" are distinct, there exists a pair of alternatives $(x, y) \in X \times X$ such that either (i) $(x, y) \in R$ and $(x, y) \notin R$" or (ii) $(x, y) \in R$" and $(x, y) \notin R$. The two cases being symmetric, we only consider the first of the two. Since $\#(R \vartriangle R') = 1$, one must have $(x', y') \in R \cap R$" for all $(x', y') \in R \cup R$" such that $(x', y') \neq (x, y)$. In order for a binary relation R' to be between R and R" as per Definition 1, one must thus have $(x', y') \in R'$ for all $(x', y') \in R \cup R$" such that $(x', y') \neq (x, y)$. If now $(x, y) \in R'$, then $R' = R$. If on the other hand $(x, y) \notin R'$, then $R' = R$". Hence, one cannot have both $R' \neq R$ and $R' \neq R$". To prove the other direction of the implication, suppose that $\#(R \vartriangle R") > 1$. This means that there are at least two distinct pairs of alternatives (x, y) and $(x', y') \in X \times X$ such that $\{(x, y), (x', y')\} \subset R \cup R$" and $\{(x, y), (x', y')\} \cap (R \cap R") = \varnothing$. We consider several cases.*
(i) $\{(x, y), (x', y')\} \subset R \backslash R$".
(ii) $\{(x, y), (x', y')\} \subset R" \backslash R$.
(iii) $(x, y) \in R \backslash R$" and $(x', y') \in R" \backslash R$
(iv) $(x, y) \in R" \backslash R$ and $(x', y') \in R \backslash R$"
'If case (i) holds, then the binary relation $R' = R" \cup \{(x, y)\}$ is distinct from both R and R" and is between them as per Definition 1. Similarly, if case (ii) holds, the binary relation $R' = R \cup \{(x, y)\}$ would be distinct from

both R and R" and between them as per Definition 1. In Case (iii), the binary relation $R' = R" \cup \{(x, y)\}$ would be distinct from R" (by containing (x, y)) and from R (by containing (x', y')), while being clearly between both R and R" as per Definition 1. Similarly for case (iv), the binary relation $R' = R \cup \{(x, y)\}$ would be distinct from R (by containing (x, y)) and from R" (by containing (x', y')), while being again between both R and R" as per Definition 1. This completes the proof. ■

Lemma 2 thus provides a simple test to check wether or not two binary relations are adjacent. Two binary relations are adjacent if and only if they differ from each other by exactly one ordered pair.

We now introduce the notion of a majoritarian preference relation associated to a given profile of such preferences. Our definition of such a notion is as follows.

Definition 4 *Given a profile of n complete and reflexive preference relations $(R_1, ..., R_n)$ on X for some integer $n \geq 2$, we say that the complete and reflexive binary relation R on X is majoritarian for $(R_1, ..., R_n)$ if it satisfies, for every x and $y \in X$, $\#\{i : (x, y) \in R_i\} > n/2 \Longrightarrow (x, y) \in R$ and $\#\{i : (x, y) \in R_{Ai}\} > n/2 \Longrightarrow (x, y) \in R_A$*

We observe that a profile of preferences $(R_1, ..., R_n)$ will typically have many such majoritarian preferences. One of them is the classical majority rule defined, for every profile of preferences $(R_1, ..., R_n)$, by $(x, y) \in R \Longleftrightarrow \#\{i : (x, y) \in R_i\} \geq n/2$. Another is the Kemeny-Young rule characterized by Young and Levenglick (1978). We record for further reference the following obvious remark concerning the definition of a Majoritarian preference.

Remark 1 *A complete and reflexive preference $R \in \mathcal{C}$ is Majoritarian with respect to the profile $(R_1, ..., R_n)$ (for some integer $n \geq 2$) if and only if it satisfies, for every x and $y \in X$, $(x, y) \in R \Longrightarrow \#\{i : (x, y) \in R_i\} \geq n/2$ and $(x, y) \notin R \Longrightarrow \#\{i : (x, y) \in R_i\} \leq n/2$.*

Proof. *In one direction, assume that $R \in \mathcal{C}$ is Majoritarian in the sense of Definition 4 for the profile $(R_1, ..., R_n) \in \mathcal{C}^n$ and let x and y be alternatives in X such that $(x, y) \in R$. Suppose by contradiction that $\#\{i : (x, y) \in R_i\} < n/2$. Since the preferences $(R_1, ..., R_n)$ are complete, this means that $\#\{i : (y, x) \in R_{A_i}\} = n - \#\{i : (x, y) \in R_i\} > n/2$. But if R is Majoritarian with respect to $(R_1, ..., R_n)$ as per definition 4, one must have $(y, x) \in R_A$, which is a contradiction. Similarly, assuming again that R is a complete and reflexive binary relation that is Majoritarian in the sense of Definition 4 with respect to the profile of preferences $(R_1, ..., R_n) \in \mathcal{C}^n$, suppose there are some x and $y \in X$ for which one has $(x, y) \notin R$. Since R is complete, one must have $(y, x) \in R_A$. But then, assuming that $\#\{i : (x, y) \in R_i\} > n/2$ would* *contradict the first requirement of Definition 4 that R is majoritarian. Hence* $\#\{i : (x, y) \in R_i\} > n/2 \leq n/2$ *must hold. The proof for the other direction is immediate.* ■

The main contribution of the paper is to characterize any majoritarian preference over some profile as a minimizer of the sum of the pairwise distances between itself and the preferences of the profile for some distance function that numerically represents a notion of pairwise dissimilarity between preferences. As it turns out, the distance-minimizing property of a majoritarian preference depends crucially upon a property of the distance that we refer to as "between-additivity". We introduce as follows this property along with a formal definition of a distance function on $\mathcal{C} \times \mathcal{C}$.

Definition 5 (Distance) *A function* $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ *is a* ***distance function*** *if it satisfies the following properties:*
(i) Non-negativity: $d(R_1, R_2) \geq 0$ *for all* $R_1, R_2 \in \mathcal{C}$.
(ii) Identity only at equality: $d(R_1, R_2) = 0$ *if and only if* $R_1 = R_2$.
(iii) Symmetry: $d(R_1, R_2) = d(R_2, R_1)$ *for all* $R_1, R_2 \in \mathcal{C}$.
(iv) Triangle Inequality: $d(R_1, R_3) \leq d(R_1, R_2) + d(R_2, R_3)$
Moreover, a function $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ *is called a* ***between-linear distance function*** *if it satisfies, in addition to (i)-(iv):*
(v) $d(R, R") = d(R, R') + d(R', R")$ *for every R, R′ and R"* $\in \mathcal{C}$ *such that* $R' \in \mathcal{B}(R, R")$

The crucial property of a distance insofar as representativeness of the majority relation goes is the property (v). This property requires the distance to be "additive" with respect to any combination of two preferences taken from three preferences that are connected by a betweenness relation. A clear implication of "between-linearity" is consistency with respect to the betweenness relation. Any preference that is between two others will always be more similar to any of these two preferences than the two preferences themselves. We state formally this as follows.

Remark 2 *Let* $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ *be a* ***between-additive distance function*** *Then, for any three distinct R, R′ and R"* $\in \mathcal{C}$ *such that* $R' \in \mathcal{B}(R, R")$*, one has:*

$$d(R, R') < d(R, R'') \text{ and } d(R', R'') < d(R, R'')$$

Proof. *Since d satisfies Triangle inequality, one has:*

$$d(R, R'') \leq d(R, R') + d(R', R")$$

The conclusion then follows at once from the fact that d satisfies non-negativity and identity at equality, and the assumption that R, R' *and* R'' *are all distinct.* ■

We now provide two well-know examples of distances between preferences, the first of which satisfying between-additivity.

Example 1 *The Kemeny notion of dissimilarity (see e.g. Kemeny (1959) and Kemeny and Snell (1962)). This distance function, denoted d^K, is defined as follows:*

$$d^K(R^1, R^2) \geq d^K(R^3, R^4) \Longleftrightarrow \#(R^1 \vartriangle R^2) \geq \#(R^3 \vartriangle R^4)$$

This distance function defines the dissimilarity between any two preferences by the number of pairs of alternatives on ranking of which the two preferences disagree. The reader can verify that this widely discussed distance is indeed a between-linear distance.

Example 2 *The Spearman (1904) notion of dissimilarity (see Monjardet (1998) for a comparison of the Kemeny and the Spearman notions of similarity among linear orderings). This distance function d^S, which only applies to orderings, is defined by:*

$$d^S(R^1, R^2) \geq d(R^3, R^4) \Longleftrightarrow [\sum_{x \in X} (r^1(x) - r^2(x))^2]^{1/2} \geq [\sum_{x \in X} (r^3(x) - r^4(x))^2]^{1/2}$$

where, for $i = 1, 2, 3, 4$, $r^i(x)$ denoted the rank of alternative x in the ordering R^i defined by:

$$r^i(x) = 1 + \#\{A \in X/R^i : (a, x) \in R_{A_i} \textit{ for } a \in A\}$$

It is readily seen that the Spearman distance is a distance function from $\mathcal{C} \times \mathcal{C}$ to the real but is not between-linear .For example if we take $X = \{a, b, c\}$ and $\widehat{R}^1 \cap = \{(a, b), (b, c), (a, c)\}$, $\widehat{R}^2 = \{(b, c), (b, a), (a, c)\}$ $\widehat{R}^3 = \{(c, b), (b, a), (c, a)\}$, it is clear that $R^2 \in \mathcal{B}(R^1, R^3)$. However:

$$\begin{aligned} d^S(R^1, R^3) &= [(3-1)^2 + (2-2)^2 + (1-3)^2]^{1/2} = 2\sqrt{2} \\ &< d^S(R^1, R^2) + d(R^2, R^3) \\ &= [(3-2)^2 + (3-2)^2]^{1/2} + [(2-1)^2 + (3-2)^2 + (3-1)^2]^{1/2} \\ &= (1 + \sqrt{3})\sqrt{2} \end{aligned}$$

We now turn to the two main results of this paper. The first one states that a preference minimizes the sum of distances between itself and a collection of preferences for some between-additive distance function if and only if this preference is majoritarian with respect to the considered collection of preference. We state formally this result as follows.

Theorem 1 *Let d be a between-additive distance function and, for some integer n, let $(R_1, ..., R_n) \in \bigcup_{n\in\mathbb{N}} C^n$ be a profile of complete preference relations. Then the complete preference $R^* \in \mathcal{C}$ satisfies the inequality*

$$\sum_{i=1}^{n} d(R_i, R^*) \leq \sum_{i=1}^{n} d(R_i, R)) \quad \forall R \in \mathcal{C}. \tag{1}$$

if and only if R^* is majoritarian for $(R_1, ..., R_n)$.

Proof.
Sufficiency. *Suppose that R^* and d are, respectively, a Majoritarian preference relation for a profile $(R_1, ..., R_n)$ for some integer $n \geq 2$ and a between-additive distance function. Consider any profile of preferences $(R_1, ..., R_n) \in \mathcal{D}$ (for some $n \in \mathbb{N}$) and any $R \in \mathcal{C}$. We need to show that $\sum_{i=1}^{n} d(R_i, R^*) \leq \sum_{i=1}^{n} d(R_i, R)$. Proving this is immediate if $R = R^*$. Hence, we assume that $R \neq R^*$. Our proof strategy is to construct a sequence of preferences $(R^0, R^1, \ldots, R^q)$ in such way that $R^0 = R$, $R^q = R^*$and the following holds:*

$$\sum_{i=1}^{n} d(R_i, R^{j+1}) \leq \sum_{i=1}^{n} d(R_i, R^j) \;\forall\;\; j \in \{0, q-1\}.$$

We construct the sequence as follows, starting with $R_0 = R$, and exploiting the fact that $R \neq R^*$.
(a) $R^j \backslash \{x^j, y^j\} = R^{j-1} \backslash \{x^j, y^j)$ and $R^j = R^{j-1} \cup \{(x^j, y^j)\}$ for some $(x^j, y^j) \in X \times X$ such that $(x^j, y^j) \in R^*$ and $(x^j, y^j) \notin R^{j-1}$ if there are such (x^j, y^j) and by:
(b) $R^j \backslash \{\overline{x}^j, \overline{y}^j\} = R^0 \backslash \{\overline{x}^j, \overline{y}^j)$ and $R^j = R^{j-1} \backslash \{(\overline{x}^j, \overline{y}^j)\}$ if there are no pair $(x, y) \in X \times X$ satisfying the condition mentioned in (a) but there are $(\overline{x}^j, \overline{y}^j) \in R^{j-1}$ such that $(\overline{x}^j, \overline{y}^j) \notin R^*$.
*Observe that this sequence is **not**, in general, unique. Indeed, at any step $t = 1, ..., q$, there can be typically many pairs either of the type mentioned in (a) or of the type mentioned in (b). But any sequence constructed in this way will do. Observe also, thanks to the alternative definition of a Majoritarian decision function provided by Remark 1, that the terminal step q of the sequence is reached when there are no pair $(x^q, y^q) \in X \times X$ such that $(x^q, y^q) \in R^*$ and $(x^q, y^q) \notin R^{q-1}$ and there are also no pair $(\overline{x}^q, \overline{y}^q) \in R^{q-1}$ such that $(\overline{x}^q, \overline{y}^q) \notin R^*$. This terminal step obviously corresponds to the situation where $R^q = R^*$. We now show that:*

$$\sum_{i=1}^{tn} d(R_i, R^j) \leq \sum_{i=1}^{n} d(R_i, R^{j-1}) \tag{2}$$

for all $j = 1, ..., q$. A preliminary step for this is the observation that if two distinct preferences R and R'' are adjacent, then for any preference

R' distinct from both R and R", one must have either $R \in \mathcal{B}(R', R")$ or $R" \in \mathcal{B}(R, R')$. To see this, we first recall that by Lemma 2, R and R'' are adjacent if and only if they differ only by one pair (say (x, y)). Suppose without loss of generality (up to a permutation of the role of R and R'' in the argument) that $(x, y) \in R$ and $(x, y) \notin R''$. Consider then any preference R' distinct from both R and R''. If $(x, y) \in R'$, then one has that $R' \cap R'' \subset R \subset R' \cup R''$ so that $R \in \mathcal{B}(R', R'')$. If on the other hand $(x, y) \notin R'$, then one has $R \cap R' \subset R'' \subset R \cup R'$ and, therefore, $R'' \in \mathcal{B}(R, R')$. We now prove Inequality (2). It is clear by the definition of the sequence given above that R^j and R^{j+1} are adjacent. They either differ by a pair (x^j, y^j) such that $(x^j, y^j) \in R^*$ and $(x^j, y^j) \notin R^{j-1}$ (Case (a)) or by a pair $(\overline{x}^j, \overline{y}^j) \in R^{j-1}$ such that $(\overline{x}^j, \overline{y}^j) \notin R^*$ (Case (b)). If we are in Case (a), then $(x^j, y^j) \in R^*$ which implies, by definition of R^* being Majoritarian (Remark 1), that $\#\{i : (x^j, y^j) \in R_i\} \geq n/2$. The observation made above about the adjacent preferences R^{j-1} and R^j and any other preference, including one observed in the profile $(R_1, ..., R_n)$ applies. In particular, for any individual i such that $x^j\ R_i\ y^j$, one has that $R_i \cap R^{j-1} \subset R^j \subset R_i \cup R^{j-1}$. Hence for any such i, $R^j \in \mathcal{B}(R_i, R^{j-1})$. We therefore have, using between-additivity of d:

$$d(R_i, R^{j-1}) = d(R_i, R^j) + d(R^{j-1}, R^j) \tag{3}$$

Analogously, for all other h (if any) such that $(x^j,\ y^j) \notin R_h$, we have that $R_h \cap R^j \subset R^{j-1} \subset R_h \cup R^j$ and, therefore, that $R^{j-1} \in \mathcal{B}(R_h, R^j)$. Using again between additivity of d, we can write:

$$d(R_h, R^j) = d(R_h, R^{j-1}) + d(R^{j-1}, R^j)$$

or:

$$d(R_h, R^{j-1}) = d(R_h, R^j) - d(R^{j-1}, R^j) \tag{4}$$

Summing Equalities (3) and (4) over all concerned individuals and rearranging yields:

$$\sum_{i:(x^j,y^j)\in R_i} d(R_i, R^{j-1}) + \sum_{h:(y^j,x^j)\in R_{Ai}} d(R_h, R^{j-1}) = \sum_{i:(x^j,y^j)\in R_i} d(R_i, R^j)$$

$$+ \sum_{h:(y^j,x^j)\in R_{Ai}} d(R_h, R^j) + [\#\{i \ : \ (x^j, y^j) \in \ R_i\} - \#\{h : (y^j, x^j) \in R_{Ah}\ \}]d(R^{j-1}, R^j)]$$

$$\geq \sum_{i:(x^j,y^j)\in R_i} d(R_i, R^j) + \sum_{h:y^j R_{Ai} y^j} d(R_h, R^j)$$

because $d(R^{j-1}, R^j) > 0$ and $\#\{i : (x^j, y^j) \in R_i\} \geq \#\{h : (y^j, x^j) R_{Ah}\ x^j\}$. If we are in Case (b), then there is a pair $(\overline{x}^j, \overline{y}^j) \in R^{j-1}$ such that $(\overline{x}^j, \overline{y}^j) \notin R^*$ and $R^j = R^{j-1} \backslash \{(\overline{x}^j, \overline{y}^j)\}$. Since $(\overline{x}^j, \overline{y}^j) \notin R^*$ and R^* is complete, one has that $\overline{y}^j\ R^*\ \overline{x}^j$. Since $R^*$is majoritarian relative to $(R_1, ..., R_n)$, we must *have - thanks to Remark 1 - that* $\#\{i : (\overline{x}^j, \overline{y}^j) \in R_i\} < n/2$. *Applying again the above reasoning on the adjacent pairs* R^j *and* R^{j-1}, *we have that* $R_h \cap R^{j-1} \subset R^j \subset R_h \cup R^{j-1}$ *for any individual* h *such that* $(\overline{x}^j, \overline{y}^j) \notin R_h$. *Hence one has* $R^j \in \mathcal{B}(R_h, R^{j-1})$ *for any such individual so that one can write, using the additivity of* d:

$$d(R_h, R^{j-1}) = d(R_h, R^j) + d(R^{j-1}, R^j) \tag{5}$$

Similarly, for any individual i *such that* $(\overline{x}^j, \overline{y}^j) \in R_i$, *we have* $R_i \cap R^j \subset R^{j-1} \subset R_i \cup R^j$ *and, therefore,* $R^{j-1} \in \mathcal{B}(R_i, R^j)$ *so that one can write for any such individual (again using the additivity of* d*):*

$$d(R_i, R^j) = d(R_i, R^{j-1}) + d(R^{j-1}, R^j)$$

or:

$$d(R_i, R^{j-1}) = d(R_i, R^j) - d(R^{j-1}, R^j) \tag{6}$$

Summing Equalities (5) and (6) over all the relevant individuals and rearranging yields:

$$\sum_{h:(\overline{y}^j,\overline{x}^j)\in R_{Ai}} d(R_h, R^{j-1}) + \sum_{i:(\overline{x}^j,\overline{y}^j)\in R_i} d(R_i, R^{j-1}) = \sum_{h:(\overline{y}^j,\overline{x}^j)\in R_{Ai}} d(R_h, R^j)$$

$$+ \sum_{i:(\overline{x}^j,\overline{y}^j)\in R_i} d(R_i, R^j) + [\#\{h : (\overline{y}^j, \overline{x}^j) \in R_{Ah}\} - \#\{i : (\overline{x}^j, \overline{y}^j) \in R_i\}] d(R^{j-1}, R^j)$$

$$\geq \sum_{h:(\overline{y}^j,\overline{x}^j)\in R_{Ah}} d(R_h, R^j) + \sum_{i:(\overline{x}^j,\overline{y}^j)\in R_i} d(R_i, R^j)$$

because $d(R^{j-1}, R^j) > 0$ *and* $\#\{h : (\overline{y}^j, \overline{x}^j) \in R_{Ah}\} \geq \#\{i : (\overline{x}^j, \overline{y}^j) \in R_i\}$. *This completes the proof that Inequality (3) holds for every* $j = 1, ..., q$. *The sufficiency part of the theorem is then proved by the transitive repetition of this inequality.*

Necessity. *Let* R^* *be a preference in* $\mathcal{C}$ *that is not majoritarian for a profile of complete preferences* $(R_1, R_2, ..., R_n)$. *This means that there are alternatives* x *and* $y \in X$ *for which either (i)* $\#\{i : (x, y) \in R_i\} > n/2$ *and* $(x, y) \notin R^*$ *or (ii)* $\#\{i : (x, y) \in R_{Ai}\} > n/2$ *and* $(x, y) \notin R^*_A$. *Suppose first that case (i) holds. Consider then any between-additive distance function* $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$. *We wish to show that there exists a preference* $R' \in \mathcal{C}$ *such that*

$$\sum_{i=1}^{n} d(R_i, R') < \sum_{i=1}^{n} d(R_i, R^*).$$

For this sake, we simply define R' *by* $R' = R^* \cup \{(x, y)\}$. *We first prove that, for any* $i \in \{1, ..., n\}$ *such that* $(x, y) \in R_i$, *one has* $R' \in \mathcal{B}(R_i, R^*)$. *Indeed, consider any* $(a, b) \in R_i \cap R^*$. *We know that* $(a, b) \neq (x, y)$ *(because* *by assumption $(x,y) \notin R^*$). Since any other pair $(a,b) \in R_i \cap R^*$ also belongs to R^*, it belongs to $R^* \cup \{(x,y)\} = R'$. Consider now any pair $(a,b) \in R'$. Either $(a,b) = (x,y)$ in which case $(x,y) \in R_i$ or $(a,b) \neq (x,y)$ in which case $(a,b) \in R^* = R' \backslash \{(x,y)\}$. Hence $R' \subset R_i \cup R^*$. We now show that for any $h \in \{1,...,n\}$ (if any) such that $(x,y) \notin R_h$, one has $R^* \in \mathcal{B}(R_h, R')$. Indeed, consider any $(a,b) \in R_h \cap R'$. Since $(x,y) \notin R_h$, we know that $(a,b) \neq (x,y)$. Hence $(a,b) \in R' \backslash \{x,y\} = R^*$. Moreover, it is clear that $R^* \subset R' \subset R' \cup R_h$. Now, exploiting the Between-additivity of d, one has*

$$\sum_{i=1}^{n} d(R_i, R') = \sum_{i:(x,y)\in R_i} d(R_i, R') + \sum_{h:(x,y)\notin R_h} [d(R_h, R^*) + d(R^*, R')] \quad (7)$$

and:

$$\sum_{i=1}^{n} d(R_i, R^*) = \sum_{h:(x,y)\notin R_h} d(R_h, R^*) + \sum_{i:(x,y)\in R_i} [d(R_i, R') + d(R', R^*)] \quad (8)$$

Subtracting (7) from (8 yields (after cancelling common terms).

$$\begin{aligned}\sum_{i=1}^{n} d(R_i, R') - \sum_{i=1}^{n} d(R_i, R^*) &= [\#\{i : (x,y) \in R_i\} - \#\{h : (x,y) \notin R_h\}] d(R', R^*) \\ &> 0\end{aligned}$$

because $\#\{i : (x,y) \in R_i\} > n/2 \geq \#\{h : (x,y) \notin R_h\}$ and $d(R', R^*) > 0$. The argument for the case (ii) is of similar nature and the details are left to the reader. ■

This theorem thus characterizes majoritarian collective decision functions as "representative" of the preferences that they take the majority of in the sense of minimizing any between-additive numerical distance between these preferences. The next theorem characterizes, somewhat dually, the property of between-additivity as being essential for the ability of majoritarian collective decision function to be representative in the sense of minimizing distance. Specifically, we prove that if a majoritarian preference for a given preference profile is to be distance-minimizing with respect to this profile for some distance function, then the distance function *must* be between-additive.

Theorem 2 *Suppose $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ is a distance function such that, for every profile $(R_1, ..., R_n) \in \mathcal{C}^n$ for some $n \geq 2$, a majoritarian preference R^* for this profile satisfies the inequality:*

$$\sum_{i=1}^{n} d(R_i, R^*) \leq \sum_{i=1}^{n} d(R_i, R) \quad \forall R \in \mathcal{C}. \quad (9)$$

Then d is between-additive.

Proof. *Suppose that a distance $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ is not between-additive. This means that there are complete binary relations R_1, R_2 and R_3 on X such that $R_2 \in \mathcal{B}(R_1, R_3)$ and*

$$d(R_1, R_3) < d(R_1, R_2) + d(R_2, R_3) \tag{10}$$

using the Triangle inequality. Consider then the profile of preferences (R_1, R_3). We first establish that R_2 is Majoritarian on the profile (R_1, R_3). To see this, we use Remark 1, and we first consider any x and y such that $(x, y) \in R_2$. Since $R_2 \in \mathcal{B}(R_1, R_3)$, one must have $(x, y) \in R_1 \cup R_3$. Hence, at least one of the two preferences (R_1, R_3) in the profile must contain the pair (x, y). Hence $\#\{i : (x, y) \in R_i\} \geq 1 = \frac{n}{2}$. Consider now x and y such that $(x, y) \notin R_2$. Since $R_2 \in \mathcal{B}(R_1, R_3)$ and, as a result, $R_1 \cap R_3 \subset R_2$, one must have that $(x, y) \notin R_1 \cap R_3$. Hence there can be at most one of the two preferences R_1 and R_3 that contains the pair (x, y). Put differently $\#\{i : (x, y) \in R_i\} \leq 1 = \frac{n}{2}$, as required by the second condition of Remark 1. Hence R_2 is Majoritarian on the profile (R_1, R_3). However, R_2 does not minimize the sum of distance between itself and the two individual preferences of the profile because, from Inequality 10 and the property of identity of the indiscernible, one has (using symmetry):

$$d(R_1, R_1) + d(R_1, R_3) = d(R_1, R_3) < d(R_1, R_2) + d(R_2, R_3) = d(R_2, R_1) + d(R_2, R_3).$$

Hence R_1 (but the argument would work just as well for R_3) has a strictly smaller aggregate distance from the individual preferences of the profile (R_1, R_3) than R_2. This completes the proof. ■

Remark 3 *With the exception of Remark 2 above, the proof of Theorem 2 is the only instance where some use is made of the Triangle inequality.*

As shown in Example 2 above, there are many plausible notions of distances between preferences that are not between additive. However, the Kemeny notion of distance is additive. One may of course wonder whether there are other notions of distance that are Between-Additive. The following example shows that there are quite a few. Hence, the results of this paper are significant generalizations of the fact that a Majoritarian preference minimizes the Kemeny distance between itself and the preferences from which it emanates.

Example 3 *Consider any function $\delta : X \times X \to [0, 1]$ be such that $\delta(x, x') = \delta(x', x)$ for all $x, x' \in X$. Functions like this clearly exist. For example, taking any linear ordering R of X, one can define δ^R by:*

$$\delta^R(x, x') = \frac{\mid r^R(x) - r^R(x') \mid}{\#X}$$

where $r^R(x)$ is the rank of x under R defined (in the case of a linear ordering) by:

$$r^R(x) = 1 + \#\{y \in X : (y, x) \in R_A\}$$

δ^R so defined obviously maps $X \times X$ into $[0, 1]$ and satisfies $\delta(x, x') = \delta(x', x)$ for all $x, x' \in X$. For any such function δ therefore, define a function $d^\delta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ as follows:

$$d^\delta(R_1, R_2) = \sum_{\{(x,x') \in R_1 \quad R_2 : x \neq x'\}} \delta(x, x').$$

It is easily verified that d^δ so defined satisfies the three first Properties of Definition 5.

- *Non-negativity: This holds by definition of δ.*
- *Identity only at equality of: For any $R_1 = R_2$ we have $(R_1 \backslash R_2) = (R_2 \backslash R_1) = \phi$. Therefore, $d^\alpha(R_1, R_1) = 0$.*
- *Symmetry: Since $(R_1 \backslash R_2) \cup (R_2 \backslash R_1) = (R_1 \backslash R_2) \cup (R_2 \backslash R_1)$ for all $R_1, R_2 \in C$, we have $d^\alpha(R_1, R_2) = d^\alpha(R_2, R_1)$.*

We now show that d^δ satisfies between-additivity. That is, for any R_1, R_2, R_3 such that $R_2 \in \mathcal{B}(R_1, R_3)$, we show that $d^\delta(R_1, R_2) + d^\delta(R_2, R_3) = d^\delta(R_1, R_3)$. From the definition of d^δ, one can write:

$$d^\delta(R_1, R_2) + d^\delta(R_2, R_3) = \sum_{(x,x') \in R_1 \triangle R_2 : x \neq x'} \delta(x, x') + \sum_{(y,y') \in R_1 \triangle R_2 : y \neq y'} \delta(y, y')$$

Since $(R \backslash R') \cup (R' \backslash R) = (R \cup R') \backslash (R \cap R')$ for all complete binary relations R and R', one can also write:

$$d^\delta(R_1, R_2) + d^\delta(R_2, R_3) = \sum_{(x,x') \in (R_1 \cup R_2) \backslash (R_1 \cap R_2) : x \neq x'} \delta(x, x') + \sum_{(y,y') \in (R_2 \cup R_3) \backslash (R_2 \cap R_3) : y \neq y'} \delta(y, y') \quad (11)$$

We now observe that the sets $(R_1 \cup R_2) \backslash (R_1 \cap R_2)$ and $(R_2 \cup R_3) \backslash (R_2 \cap R_3)$ are disjoint. Indeed, suppose that $(x, x') \in (R_1 \cup R_2) \backslash (R_1 \cap R_2)$. Then, either (i) $(x, x') \in R_1 \backslash R_2$ or (ii) $(x, x') \in R_2 \backslash R_1$. In case (i), we know that $(x, x') \notin R_3 \backslash R_2$ (by definition of $R_2 \in \mathcal{B}(R_1, R_3)$). Since by assumption $(x, x') \notin R_2$, one has $(x, x') \notin R_2 \backslash R_3$. Hence $(x, x') \notin R_2 \backslash R_3 \cup R_3 \backslash R_2 = (R_2 \cup R_3) \backslash (R_2 \cap R_3)$. In case (ii), we know by definition that $(x, x') \notin R_3 / R_2$ (because $(x, x') \in R_2$). Since $R_2 \in \mathcal{B}(R_1, R_3)$, one can not have $(x, x') \in R_2 \backslash R_3$ (because $R_2 \subset R_1 \cup R_3$). Hence any pair in the set $(R_1 \cup R_2) \backslash (R_1 \cap R_2)$ is not in the set $(R_2 \cup R_3) \backslash (R_2 \cap R_3)$ so that the two sets are disjoint. We now show that

$$[(R_1 \cup R_2) \backslash (R_1 \cap R_2)] \cup [(R_2 \cup R_3) \backslash (R_2 \cap R_3)] = (R_1 \cup R_3) \backslash (R_1 \cap R_3)$$

We first prove that $[(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)] \subset (R_1 \cup R_3)\backslash(R_1 \cap R_3)$*. Consider for this sake any pair of alternatives* $(x, x') \in [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$*. Four (non-mutually exclusive) cases are compatible with this consideration:*
(i) $(x, x') \in R_1\backslash R_2$
(ii) $(x, x') \in R_2\backslash R_1$
(iii) $(x, x') \in R_2\backslash R_3$
(iv) $(x, x') \in R_3\backslash R_2$
Consider Case (i). Since $R_2 \in \mathcal{B}(R_1, R_3)$*, one can not have* $(x, x') \in R_3$ *(because in this case, one would have* $(x, x') \in R_1 \cap R_3 \subset R_2$*, in contradiction of* $(x, x') \in R_1\backslash R_2$*). Hence* $(x, x') \in R_1\backslash R_3 \subset (R_1 \cup R_3)\backslash(R_1 \cap R_3)$*. Suppose now that we are in Case (ii). By assumption* $(x, x') \notin R_1$ *and* $(x, x') \in R_2 \subset R_1 \cup R_3$ *(since* $R_2 \in \mathcal{B}(R_1, R_3)$*). Hence* $(x, x') \in R_3\backslash R_1 \subset (R_1 \cup R_3)\backslash(R_1 \cap R_3)$*. For Cases (iii) and (iv), we just apply the argument of case (ii) and (i) (respectively) up to permuting* R_1 *and* R_3*. We now prove that* $(R_1 \cup R_3)\backslash(R_1 \cap R_3) \subset [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$*. Let* $(x, x') \in (R_1 \cup R_3)\backslash(R_1 \cap R_3)$*. This means either that* $(x, x') \in R_1\backslash R_3$ *or that* $(x, x') \in R_3\backslash R_1$*. In the first case either* $(x, x') \in R_2$ *(in which case* $(x, x') \in R_2\backslash R_3 \subset [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$*) or* $(x, x') \notin R_2$ *(in which case* $(x, x') \in R_1\backslash R_2 \subset [(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)]$*). The argument for the other case is similar. Since the sets* $(R_1 \cup R_2)\backslash(R_1 \cap R_2)$ *and* $(R_2 \cup R_3)\backslash(R_2 \cap R_3)$ *are disjoint and are such that*

$$[(R_1 \cup R_2)\backslash(R_1 \cap R_2)] \cup [(R_2 \cup R_3)\backslash(R_2 \cap R_3)] = (R_1 \cup R_3)\backslash(R_1 \cap R_3)$$

one can write Equality (11) as:

$$\begin{aligned} d^\delta(R_1, R_2) + d^\delta(R_2, R_3) &= \sum_{(x,x') \in (R_1 \cup R_2)\backslash(R_1 \cap R_2): x \neq x'} \delta(x, x') + \sum_{(y,y') \in (R_2 \cup R_3)\backslash(R_2 \cap R_3): y \neq y'} \delta(y, y') \\ &= \sum_{(x,x') \in (R_1 \cup R_3)\backslash(R_1 \cap R_3): x \neq x'} \delta(x, x') \\ &= d^\delta(R_1, R_3) \end{aligned}$$

as required by between-addivity.

3 Properties of notions of preference dissimilarity that give rise to between-additive distances.

It is of interest to identify the properties of qualitative notions of preferences dissimilarity that can be numerically represented by between-additive functions. In this section, we provide an imperfect attempt in this direction.

Our attempt starts with an underlying notion of dissimilarity taking the form of a *quaternary* relation Q on $\mathcal{C}$ or, alternatively, a binary relation on

$\mathcal{C} \times \mathcal{C}$. We interpret the statement $(R^1, R^2)\ Q\ (R^3, R^4)$ as meaning that preference R^1 is weakly more dissimilar from R^2 than R^3 is from R^4. A corresponding interpretation is given to the comparative statements made with Q_A (strictly more dissimilar) and Q_S (equally dissimilar). We restrict attention to primitive notions of dissimilarity that satisfy the following axioms

Axiom 1 ***Ordering***. *Q is an ordering of $\mathcal{C} \times \mathcal{C}$.*

Axiom 2 ***Symmetry.*** *For all $R, R' \in \mathcal{C}$, $(R, R')\ Q\ (R', R)$.*

Axiom 3 ***Strict recording of distinctiveness***. *For all distinct $R, R' \in \mathcal{C}$, $(R, R')\ Q_A\ (R, R)$.*

Axiom 4 ***Perfect similarity for identical preferences.*** *For all $R, R' \in \mathcal{C}$, $(R, R)\ Q\ (R', R')$.*

Axiom 5 ***Segmental Betweenness Consistency***. *For all preferences $R^1, R^2, R^3, \overline{R}^1, \overline{R}^2$ and $\overline{R}^3 \in \mathcal{C}$ such that $R^2 \in \mathcal{B}(R^1, R^3)$ and $\overline{R}^2 \in \mathcal{B}(\overline{R}^1, \overline{R}^3)$, one must have:*
(i) $(R^1, R^2)\ Q\ (\overline{R}^1, \overline{R}^2)$ and $(\overline{R}^1, \overline{R}^3)\ Q\ (R^1, R^3) \Longrightarrow (\overline{R}^2, \overline{R}^3)\ Q\ (R^2, R^3)$ and
(ii) $(R^1, R^2)\ Q_A\ (\overline{R}^1, \overline{R}^2)$ and $(\overline{R}^1, \overline{R}^3)\ Q\ (R^1, R^3)$ or $(R^1, R^2)\ Q\ (\overline{R}^1, \overline{R}^2)$ and $(\overline{R}^1, \overline{R}^3)\ Q_A\ (R^1, R^3) \Longrightarrow (\overline{R}^2, \overline{R}^3)\ Q_A\ (R^2, R^3)$

The properties captured by Axioms 1-4 are quite intuitive when applied to a notion of qualitative dissimilarity between preferences (or for that matter to any objects). The ordering Axiom just says that a dissimilarity comparative statement can be made for any two pairs of preferences. It also requires, through transitivity, that these comparative statements be consistent with each other. The symmetry requirement is also natural when applied to statements about dissimilarities of objects. A bit strong, but nonetheless natural, is also the requirement for two distinct preferences to be strictly more dissimilar than any one of the two preferences duplicated. There is indeed a strong presumption that there would be no-dissimilarity whatsoever between one preference and itself. Axiom 4 makes this presumption formal.

The only axiom that is worth discussing a bit more is the requirement that dissimilarity statements about pairs of preferences be "segmentally consistent" with the notion of betweenness provided by Definition 1. It basically imposes that the dissimilarity from any two preferences one the one hand and any preference that is between the two be endowed with a "segmental looking" structure. Consider indeed two preferences, and consider a preference that is between these two. One can view these three preferences as lying on some segment, the end point of which being the two extreme preferences, and the middle point being the intermediate preference. Axiom 5 requires that all such segments made of three preferences, one lying between the two others be ordered by the quaternary relation in a way that respect their "segmental" nature. That is, if one segment of three preferences is longer than another, then it is impossible to have an opposite rankings of all pairs of sub-segments of the two segments. This is, in substance, what Axiom 5 requires. It is important to observe that, among other things, Axiom 5 requires the quaternary relation Q to be consistent with betweenness in the sense that the dissimilarity between any two distinct preferences R^1 and R^2 be strictly larger than the dissimilarity between either of R^1 or R^2 and any preference that lies strictly between them. We state this formally as follows.

Lemma 3 *If Q is a quaternary relation on $\mathcal{C}$ that satisfies Axioms 1 - 5, then, for any preferences R^1, R^2 and R^3 such that $R^2 \in \overline{\mathcal{B}(R^1, R^3)}$, one has $(R^1, R^3)\ Q_A\ (R^1, R^2)$ and $(R^1, R^3)\ Q_A\ (R^2, R^3)$.*

Proof. *Suppose that R^1, R^2 and R^3 are three preferences such that $R^2 \in \overline{\mathcal{B}(R^1, R^3)}$. We only show that $(R^1, R^3)\ Q_A\ (R^1, R^2)$ must hold (the argument being similar for $(R^1, R^3)\ Q_A\ (R^2, R^3)$). Assume by contradiction that $(R^1, R^3)\ Q_A\ (R^1, R^2)$ does not hold. Since by Axiom 1 Q (as a binary relation on $\mathcal{C} \times \mathcal{C}$) is, one must have $(R^1, R^2)\ Q\ (R^1, R^3)$. If we now apply the Clause (i) of Axiom 5 to the case where the preferences $\overline{R}^1, \overline{R}^2$ and $\overline{R}^3$ mentioned in this Axiom are, respectively, R^1, R^2 and R^2, one concludes that $(R^2, R^2)\ Q\ (R^2, R^3)$. But this is contradicts Axiom 3 since, by Definition of strict betweenness, R^2 and R^3 are distinct.* ■

As an ordering on the set $\mathcal{C} \times \mathcal{C}$, a quaternary relation Q can be numerically represented by a function $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ having the property that $(R^1, R^2)\ Q\ (R^3, R^4) \iff d(R^1, R^2) \geq d(R^3, R^4)$ for any two pairs (R^1, R^2) and (R^3, R^4) of complete binary relations. It is not difficult to see (see e.g. Krantz, Luce, Suppes, and Tversky (1971), Vol. 2, ch. 14, Theorem 1) that the function d can be chosen to take positive real values and be such that $d(R^1, R^2) = 0$ if and only if $R^1 = R^2$. Moreover, since it numerically represents a symmetric quaternary relation Q by Axiom 2, d will be a symmetric function as well. As shown in Theorem 14 of Vol 2 of Krantz, Luce, Suppes, and Tversky (1971), we can also without loss of generality requires d to satisfy the so-called "triangle inequality" that $d(R, R") \leq d(R, R') + d(R', R")$ for any three preferences R, R' and R''. However, as noticed in Remark 3, the only role played by the Triangle inequality in this paper is to establish the necessity of the between-additivity of the distance function that a majoritarian preference systematically minimizes.

However, while Axioms 1-5 are clearly necessary for admitting a numerical representation that is between-additive, they are not sufficient for that purpose. We now provide a set of sufficient conditions for the desired numerical representation.

We do this partly by imposing structural properties on the quaternary relation Q that guarantee the possibility of segmentally measuring dissimilarity along the lines introduced already by Axiom 5. Imagine in effect that the comparative dissimilarities of two pairs of preferences, say (R^1, R^2) and (R^3, R^4), could be matched exactly by the dissimilarity between two preferences and some preference lying between them. Specifically, suppose that, for the preferences R^1, R^2, R^3 and R^4, there are preferences R, R' and R'' such that $R'' \in B(R, R')$, $(R, R")\ Q_S\ (R^1, R^2)$ and $(R", R')\ Q_S\ (R^3, R^4)$. This means that the dissimilarities of the pairs (R^1, R^2) and (R^3, R^4) are measured by two "adjacent" segments along a "line " connecting R and R' and passing trough some intermediate preference R''. The property we are about to introduce imposes some existential restrictions on the possibility of linearly measuring in this fashion dissimilarity. It does not assume that this possibility exists for any two pairs of preferences. But it does require that when this possibility exists for specific dissimilarity comparisons, it also exists for others. Before introducing this property, we formally define what we mean by linear dissimilarity measurement.

Definition 6 *The dissimilarity of two pairs of preferences* (R^1, R^2) *and* (R^3, R^4) *is said to be compared along the line* $(R, R")$*, which we denote by* $(R^1, R^2) \circ^L (R^3, R^4) = (R, R")$*, if there exists some* $R' \in \mathcal{B}(R, R")$ *with* $R' \neq R$ *and* $R' \neq R''$ *such that* $(R, R')\ Q_S\ (R^1, R^2)$ *and* $(R', R")\ Q_S\ (R^3, R^4)$*.*

From a formal point of view, the possibility of comparing the dissimilarity of two pairs of preferences along a line defines a binary operation $\circ^L$ on the set $\mathcal{C} \times \mathcal{C}$ or, equivalently, a function from $(\mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C})$ to $\mathcal{C} \times \mathcal{C}$. This function is not empty because there are many pairs of preferences (R^1, R^2) and (R^3, R^4) whose dissimilarity can be compared in this fashion. In effect, and trivially, any pair of preferences (R^1, R^2) and (R^3, R^4) such that $R^2 = R^3 \in \mathcal{B}(R^1, R^4)$ has this property. However the binary operation $\circ^L$ is *not* defined for all pairs of preferences in R $\times\ \mathcal{C}$. Indeed, since X is finite, there are only finitely many pairs in $\mathcal{C} \times \mathcal{C}$. One of these pairs - $(\widehat{R}, \widehat{R}")$ say - is therefore maximally dissimilar in the sense that $(\widehat{R}, \widehat{R}")\ Q\ (R, R')$ for any two preferences R and R' in $\mathcal{C}$. It would then be clearly impossible with a quaternary relation Q satisfying Axioms 1-5 to compare the dissimilarity of $(\widehat{R}, \widehat{R}")$ and any pair (R^3, R^4) of two distinct preferences along some line $(R, R")$. Indeed, suppose that such a comparison was possible. This would imply the existence of a pair $(R, R")$ of distinct preferences such that $(\widehat{R}, \widehat{R}")$ $Q_S\ (R, R')$ and $(R^3, R^4)\ Q_S\ (R', R")$ for some preference $R' \in \mathcal{B}(R, R")$. But since $(\widehat{R}, \widehat{R}")$ is maximally dissimilar, one must have $(\widehat{R}, \widehat{R}")\ Q\ (R, R")$. If Q satisfies Axiom 5, then one has by Lemma 3 that $(R, R")\ Q\ (R, R')\ Q_S$

$(\widehat{R}, \widehat{R}")$. It would then follow from transitivity that (R, R') Q_S $(R, R")$. But this can only happens by Lemma 3 if $R' = R$. But this, given Axioms 3 and 4, contradicts the fact that $(\widehat{R}, \widehat{R}")$ Q_S (R, R') and that $(\widehat{R}, \widehat{R}")$ is the most dissimilar pair in R × R.

Since the binary operation $\circ^L$ is not defined on the whole set $(\mathcal{C} \times \mathcal{C}) \times (\mathcal{C} \times \mathcal{C})$, we denote by $\mathcal{D}^{\circ^L}$ the domain of definition of $\circ^L$. Hence, $\mathcal{D}^{\circ^L} = \{(R^1, R^2, R^3, R^4) \in (\mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}) : \exists\ R,\ R'$ and $R" \in \mathcal{C}$ satisfying $R' \in \mathcal{B}(R, R")$ such that (R^1, R^2) Q_S (R, R') and (R^3, R^4) Q_S $(R', R")\}$.

We now impose three conditions on the quaternary relation Q which ensure that the set of pairs of preferences that can be compared along a line - in the sense of the binary operation $\circ^L$- is sufficiently rich. The first condition is existential. It is stated as follows.

Condition 1 *For any preferences R^1, R^2, R^3, R^4 for which there are preferences R, R' and R'' such that $R' \in \mathcal{B}(R, R'')$, (R, R') Q_S (R^1, R^2) and (R', R'') Q_S (R^3, R^4), if (R^1, R^2) Q (R^5, R^6) for some preferences R^5 and R^6, then there are preferences $\overline{R}$, $\overline{R}'$ and $\overline{R}''$ such that $\overline{R}' \in \mathcal{B}(\overline{R}, \overline{R}'')$, $(\overline{R}, \overline{R}')$ Q_S (R^5, R^6) and $(\overline{R}', \overline{R}'')$ Q_S (R^3, R^4).*

In words, this condition just requires that if two pairs of preferences (R^1, R^2) and (R^3, R^4) can be compared - on their basis of their relative dissimilarity - along some line, then so can the pairs of preferences (R^5, R^6) and (R^3, R^4) for any pair of preferences (R^5, R^6) that are not strictly more dissimilar than (R^1, R^2).

The next condition is also existential. It is stated as follows.

Condition 2 *For any preferences R^1, R^2, R^3, R^4 such that (R^1, R^2) Q_A (R^3,R^4), there must exist preferences R^5 and R^6 for which there are preferences $\overline{R}$, $\overline{R}'$ and $\overline{R}''$ such that $\overline{R}' \in \mathcal{B}(\overline{R}, \overline{R}'')$, $(\overline{R}, \overline{R}')$ Q_S (R^3, R^4), $(\overline{R}', \overline{R}'')$ Q_S (R^5, R^6) and (R^1, R^2) Q $(\overline{R}, \overline{R}'')$.*

In plain English, this condition say that if two preferences (R^1, R^2) are strictly more dissimilar than (R^3, R^4), then one can find a pair of preferences R^5 and R^6 that can be compared with (R^3, R^4) along some line $(\overline{R}, \overline{R}")$ whose endpoints are not strictly more dissimilar than (R^1, R^2). In short, if the preferences (R^1, R^2) are strictly more dissimilar than (R^3, R^4), then there is a pair of preferences R^5 and R^6 whose dissimilarity, when "added" to that of (R^3, R^4) along some segment $(\overline{R}, \overline{R}'')$, would still preserve the greater dissimilarity of (R^1, R^2) vis-à-vis the end point of the line segment $(\overline{R}, \overline{R}")$.

Condition 3 *For any preferences R^1, R^2, R^3, R^4 for which there are preferences R, R' and R'' such that $R' \in \mathcal{B}(R, R'')$, (R, R') Q_S (R^1, R^2) and (R', R'') Q_S (R^3, R^4), if R^5 and R^6 are preferences for which there are preferences $\widehat{R}$, $\widehat{R}'$ and $\widehat{R}''$ such that $\widehat{R}' \in B(\widehat{R}, \widehat{R}'')$, $(\widehat{R}, \widehat{R}')$ Q_S (R, R'') and*

$(\widehat{R}', \widehat{R}'')$ Q_S (R^5, R^6)*, then there must be preferences* $\widetilde{R}$*,* $\widetilde{R}'$ *and* $\widetilde{R}''$ *and* $\overline{R}$*,* $\overline{R}'$ *and* $\overline{R}''$ *satisfying* $\widetilde{R}' \in \mathcal{B}(\widetilde{R}, \widetilde{R}'')$*,* $\overline{R}' \in B(\overline{R}, \overline{R}'')$ *such that* $(\widetilde{R}, \widetilde{R}')$ Q_S (R^3, R^4)*,* $(\widetilde{R}', \widetilde{R}'')$ Q_S (R^5, R^6)*,* $(\overline{R}, \overline{R}')$ Q_S (R^1, R^2) *and* $(\overline{R}', \overline{R}'')$ Q_S $(\widetilde{R}, \widetilde{R}'')$*. Moreover, one must have that* $(\widehat{R}, \widehat{R}'')$ Q_S $(\overline{R}, \overline{R}'')$*.*

In words, this condition imposes some consistency in the possibilities of measuring the dissimilarities of the three pairs of preferences (R^1, R^2), (R^3, R^4) and (R^5, R^6) sequentially. If it is possible to measure these three pairs first along the line (R, R'') (for (R^1, R^2) and (R^3, R^4)) and then along the line $(\widehat{R}, \widehat{R}'')$ (for (R, R'') and (R^3, R^4) and it is also possible to measure those thee same pairs first along the line $(\widetilde{R}, \widetilde{R}'')$ (for (R^3, R^4) and (R^5, R^6)), and then along the line $(\overline{R}, \overline{R}'')$ (for (R^1, R^2) and $(\widetilde{R}, \widetilde{R}'')$, then the end points of the two lines along which the sequential measurement procedure has been performed (namely $(\widehat{R}, \widehat{R}'')$ and $(\overline{R}, \overline{R}''))$ should be equally dissimilar.

As it happens, a preference similarity quaternary relation Q satisfying these three conditions along with Axioms 1 - 5 can be numerically represented by a Between-additive distance function that may or may not satisfy the Triangle inequality. We establish this in the following theorem.

Theorem 3 *Let* Q *be a quaternary relation on* X *(or a binary relation on* $X \times X$*) that satisfies Axioms 1 - 5 and Conditions 1-3. Then, there exists a between-additive function* $d : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}_+$ *satisfying Properties (i)-(iii) of Definition 5 (but not necessarily the Triangle inequality) such that, for any four complete binary relations* R_1*,* R_2*,* R_3 *and* R_4 *on* X*, one has* (R_1, R_2) Q $(R_3, R_4) \Longleftrightarrow d(R_1, R_2) \geq d(R_3, R_4)$*.*

Proof. *The proof rides (heavily) on Theorem 3 at p. 85 of Vol 1 of Krantz, Luce, Suppes, and Tversky (1971). We first show that the quadruple* $(\mathcal{C} \times \mathcal{C}, Q, \mathcal{D}^{\circ^L} \circ^L)$ *is what these authors call (Definition 3, p. 84) an extensive structure with no essential maximum. This amount to show that:*
(1) Q *is an ordering of* $\mathcal{C} \times \mathcal{C}$ *(which it is!).*
(2) If $(R^1, R^2, R^3, R^4) \in \mathcal{D}^{\circ^L}$ *and* $((R^1, R^2) \circ^L (R^3, R^4), R^5, R^6) \in \mathcal{D}^{\circ^L}$ *then* $(R^3, R^4, R^5, R^6) \in \mathcal{D}^{\circ^L}$*,* $(R^1, R^2, (R^3, R^4) \circ^L (R^5, R^6)) \in \mathcal{D}^{\circ^L}$ *and* $((R^1, R^2) \circ^L (R^3, R^4), R^5, R^6)$ Q_S $(R^1, R^2, (R^3, R^4) \circ^L (R^5, R^6))$*. This property is an immediate consequence of Condition 3.*
(3) lf $(R^1, R^2, R^3, R^4) \in \mathcal{D}^{\circ^L}$ *and if* (R^1, R^2) Q (R^5, R^6) *for some preferences* R^5 *and* R^6*, then* $(R^3, R^4, R^5, R^6) \in \mathcal{D}^{\circ^L}$ *and* $(R^1, R^2) \circ^L (R^3, R^4)$ Q $(R^3, R^4) \circ^L (R^5, R^6)$*. The first part of this property is an immediate consequence of Condition 1. The second part of the property results at once from Segmental consistency applied to the pairs of preferences* $(R^1, R^2) \circ^L (R^3, R^4)$ *and* $(R^3, R^4) \circ^L (R^5, R^6)$*.*
(4) If R^1*,* R^2*,* R^3 *and* R^4 *are preferences for which* (R^1, R^2) Q_A (R^3, R^4)*, then there must exist preferences* R^5 *and* R^6 *such that* $(R^3, R^4, R^5, R^6) \in \mathcal{D}^{\circ^L}$ *and such that* (R^1, R^2) Q $((R^3, R^4) \circ^L (R^5, R^6))$*. This is secured by*

Condition 2.
(5) If $(R^1, R^2, R^3, R^4) \in \mathcal{D}^{\circ^L}$*, then* $(R^1, R^2) \circ^L (R^3, R^4)\ Q_A\ (R^1, R^2)$*. This is an immediate consequence of the definition of* $\circ^L$ *and Lemma 3.*
(6) If a sequence $(R, R')^n$ *(for* $n = 1, ...,$ *and two preferences* R *and* R'*) can be recursively defined by* $(R, R')^1 = (R, R') \circ^L (R, R')$ *and* $(R, R')^n = (R, R')^{n-1}$ *for any* $n = 2, ...$ *and is such that there exists a pair of preferences* $\overline{R}$ *and* $\overline{R}'$ *such that* $(\overline{R}, \overline{R}')\ Q\ (R, R')^n$ *for any* n *in the sequence, then this sequence should be finite. This (Archimedean) axiom is satisfied here in our finite setting.*
Hence, using Theorem 3 at p. 85 of Vol 1 of Krantz, Luce, Suppes, and Tversky (1971), we conclude that there exists a function $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ *such that* $(R_1, R_2)\ Q\ (R_3, R_4) \iff d(R_1, R_2) \geq d(R_3, R_4)$ *for any four complete binary relations* R_1*,* R_2*,* R_3 *and* R_4 *on* X*. The function* d *inherits the properties of* Q *and satisfies therefore Properties (i)-(iii) of Definition 5. We know from the very same Theorem 3 of Krantz, Luce, Suppes, and Tversky (1971) that* $d((R_1, R_2) \circ^L (R_3, R_4)) = d(R_1, R_2) + (R_3, R_4)$ *for all preferences* $(R^1, R^2, R^3, R^4) \in \mathcal{D}^{\circ^L}$*. Since any three preferences* R_1, R_2 *and* R_3 *such that* $R_2 \in \mathcal{B}(R_1, R_3)$ *are obviously also such that* $(R^1, R^2, R^2, R^3) \in \mathcal{D}^{\circ^L}$*, this also shows that* d *is between-additive.* ■

While Theorem 3 does not guarantee a numerical representation of a qualitative notion of preference dissimilarity through a between-additive distance function satisfying the Triangle inequality, it does guarantees the between-additivity of the function d. Since the Triangle inequality plays no role in proving that a preference that is majoritarian with respect to some preferences profile minimizes the sum of distances between itself and the preferences in the profile, one can say that a Majoritarian preference minimizes the sum a numerical representation of a qualitative notion of preferences dissimilarities between itself and any profile of preference under any notion of preference dissimilarity satisfying Axioms 1 - 5 and Conditions 1-3. This, we believe, adds some generality to the analysis of this paper.

4 Conclusion

This paper has provided what we believe to be a significant generalization of a relatively little known argument in favour of the "preference of the majority" for collective decision making. We have shown, in effect, that the preference of the majority is representative of the collection of preferences from which it emanates in the sense of minimizing the aggregate pairwise dissimilarity between those preferences and itself for a reasonably general notion of such pairwise dissimilarity. This property of the majoritarian rule was known with respect to the Kemeny notion of distance in the case where the preference of the majority is transitive. We have shown that the property holds true for a larger class of notions of preference dissimilarity, and for preferences that do not need to be transitive. We have also identified a property - between-additivity - of a distance function representing the underlying notion of dissimilarity that is necessary and sufficient for majority to be representative in this sense sense. We have also provided an unsatisfactory characterization of an ordinal notion of preference dissimilarity that can be numerically represented by such a between-additive distance function. Our characterization is unsatisfactory for at least two reasons. First, it rides on three unnecessary structural assumptions that may be difficult to verify in practice. Second, our characterization does not guarantee the possibility of representing the ordinal notion of preference dissimilarity by a function satisfying the triangle inequality. While the Triangle inequality plays no role in the result that majoritarian preferences are the only ones that minimize a between-additive distance between themselves and the preferences from which they emanate, it does play a role for showing that between-additivity of the distance is necessary for being minimized by a majoritarian rule. We therefore believe that obtaining a more satisfactory characterization of a notion of preference dissimilarity that is numerically representable by a between-additive distance is a well-worth objective for future research.

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