Distributional considerations during growth toward the golden rule

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January 30, 2019

Abstract

In a productive economy, savings made by a generation are expected to benefit more than proportionally to future generations. But how such a sacrifice for growth should be shared among heterogeneous regions? With constant elasticity of substitution specification, I show that there is an unequal 'burden-sharing' between North and South when the economy grows toward the highest sustainable level of welfare: the golden rule. This depends upon two fundamentals elements: the marginal productivity gap and the inequality aversions. The intragenerational inequality aversion does not impact the final steady state. But the intra and intergenerational inequality aversions impact the dynamic. In particular, the higher the inequality aversions, the more North and South share an equal burden to grow. Unequal treatments do not allow to reverse the higher sacrifice asked to South. This paper leads to the conclusion that transfers from North to South shall be implemented for a sustainable development.

Keywords: renewable resource, intragenerational equity, intergenerational equity, Ramsey model, heterogeneous capital

JEL Classification: D63, O44, Q20, Q56

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†I am grateful to Vincent Martinet for his valuable comments and suggestions.

> "...we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination."
>
> Ramsey (1928, p. 543)

Abstract

1 Introduction

Growth rather than decline is surely an unanimous criterion of economic development. At least as long as this process is sustainable. But for making a production (then implicitly an asset) to grow, we shall not consume our entire income, i.e. "*the amount which [people] can consume without impoverishing themselves*" (Hicks, 1939/1946, p. 172).[1] For a natural resource, we shall not harvest the entire renewal. In a word, we need savings. In an intergenerational perspective, as conceived by Ramsey (1928), this can be viewed as a sacrifice of the current generation to the benefit of future ones. But how should this sacrifice be shared among individuals living in the same generation? Growth theory already answered this question for an aggregate capital and a representative agent through the optimal savings timing. But the question of sharing sacrifice comes to interest once there is an heterogeneity. As argued by Schelling (1995): "*it is this willingness to model all humankind as a single agent that makes optimization models attractive, feasible, and inappropriate*".

The one-sector Solow-Swan (1956) neoclassical growth model was extended to the two-sector case by Uzawa (1961, 1963), but the savings rate is considered as exogenous. The Ramsey model of optimal saving was extended to the multiple goods-sectors case by Samuelson and Solow (1956). But their approach was not very much followed in the literature. A notable exception that does not collapse to the single sector case was provided by Pitchford (1977). But the two sectors, each in one region of the economy, produce one single good, which does not allow to consider inequality in consumptions. Rather, two-sector economy was mainly analyzed through the discounted version of the Ramsey model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) (RCK hereafter). This was the case to deal with, for instance, physical capital and exhaustible resources (Dasgupta and Heal, 1974) or physical and human capital (Lucas, 1988).[2] On the other hand, heterogeneity of agents was analyzed both in the Solow-Swan model (Stiglitz, 1969) and in the RCK model, through different individual discount rates

[1] Hicksian income refers more precisely to "*the maximum value which [a man] can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning. Thus, when a person saves, he plans to be better off in the future; when he lives beyond his income, he plans to be worse off*" (p. 172). This idea can be tracked back to Fisher (1906) and Lindahl (1933) (cited in Heal, 1998, p. 156).

[2] In a sense, a two-sector RCK model was earlier proposed by Srinivasan (1964) and Uzawa (1964), but in an non-utilitarian approach.

(Ramsey, 1928; Becker, 1980; Becker and Foias, 1987) or through different initial endowments of wealth (Chatterjee, 1994; Caselli and Ventura, 2000)[3]. Heterogeneity of both sectors and agents in the RCK model was proposed by Becker and Tsyganov (2002), but there are one capital good and one production good that can be aggregated into one single production. Endress et al. (2014) combined intergenerational equity with individual impatience in an overlapping generations model, but without referring to inequality aversion.

More broadly, both intra and intergenerational considerations of sharing resources can be found in the growth-inequality literature. Even if trends of long-term economic development and allocation of wealth was analyzed by the classical economists in the Nineteenth century, it was first extensively analyzed with use of data by Kuznets (Kuznets and Jenks, 1953; Kuznets, 1955). According to him, inequalities rise, then decrease, in the process of development. This topic was further analyzed by Atkinson and Harrison (1978), and current works refute the Kuznets hypothesis (e.g. Piketty, 2015). For a review of the growth-inequality relationship see Aghion et al. (1999) and Garcìa-Peñalosa (2017).

Though linked, the question here is different in the sense that I am interested in linking ethical judgments with outcomes of the allocation, both between individuals living at the same time and between different generation living, by definition, at different dates. This is the so-called intra and intergenerational equity relationship (Isaac and Piacquadio, 2015, Kverndokk et al., 2014, and the references in there). Equity can be taken as a requirement of equal treatment. Here, all individuals (and all generations between them) will have the same weight in the public decision[4].

On the intergenerational dimension, the intertemporal social welfare criterion has to satisfy finite anonymity: a finite number of permutations in the time of appearance of two generations shall not change the result. This is supported by a long tradition in economics and philosophy against discounting future welfares. For Sidgwick (1874/1962, p. 414) "*the time at which a man exists cannot affect the value of his happiness from a universal point of view*". Ramsey (1928, p. 543) qualified discounting as "*a practice which is ethically indefensible and arises merely from the weakness of the imagination*". Pigou (1920, p. 25) argued that "*our telescopic faculty is defective*", and suggested that "*the State should protect the interests of the future* in some degree *against the effects of our irrational discounting and of our preference for ourselves over our descendants* (Pigou, 1932, p. 29). For Harrod (1948, p. 40), the pure time preference is a "*polite expression for rapacity and the conquest of reason by passion*". For Rawls (1971, p. 287), "*from a moral point of view there are no grounds for discounting future well-being on the basis of pure time preference*". Finally, according to Solow (1993, p. 165) "*no generation*

[3]See also Garcìa-Peñalosa and Turnovsky (2012) and the references in there.

[4]There is not a unique accepted definition of the word 'equity'. Our requirement here is in line with a dictionary definition: "equity is the quality of being fair and reasonable in a way that gives equal treatment to everyone." (source: `https://www.collinsdictionary.com/dictionary/english/equity`, visited on May 2018).

'should' be favored over any other". But as shown by Koopmans (1960), if a (social) criterion of infinite utility stream satisfies some[5] *a priori* desirable axioms, it has to exhibit 'impatience', i.e. discounting. Following his approach, Diamond (1965) stated a classic impossibility result: such a criterion cannot both be efficient (strong Pareto) and treat all generations equally. For a critical survey of related results see Asheim (2010). The Ramsey (1928) approach satisfies both 'efficiency' and 'equity' but is incomplete.

The intragenerational equity can be understood in several ways, see for example Fleurbaey and Maniquet (2011). But, far from reaching exhaustiveness, a *symmetric* social welfare function *à la* Bergson (1938)-Samuelson (1947) can represent different equity views of a society.

Here I study savings – understood as the difference between the sustainable individual maximal consumption and the actual consumption – made by different individuals when society wishes to reach the highest sustainable social well-being: the golden rule (Ayong Le Kama, 2001). This may be interpreted as 'generosity' toward future possible attainable welfare (Gerlagh, 2017). I model two regions, North and South, each one has access to a different resource stock, in the spirit of Samuelson and Solow (1956). The intratemporal social preferences are represented through a social welfare function (SWF) with a constant elasticity of substitution (CES). The parameter of elasticity may be interpreted as a parameter of inequality aversion (Atkinson, 1970). Indeed, such a function encompasses two famous special cases: utilitarianism (elasticity goes to infinity) and the symmetric minimum, sometimes called "Rawlsian" (elasticity goes to zero). Symmetrically, the intertemporal social preferences are represented through an intertemporal SWF with a constant intertemporal elasticity of substitution (CIES). More precisely, I minimize the difference between the welfare targeted and the actual welfare, in the spirit of Ramsey (1928). Here also, the parameter of intertemporal elasticity may be interpreted as a parameter of intertemporal inequality aversion (IA). As in its instantaneous counterpart, such a function can tend to a nil IA case (elasticity goes to infinity) or to an infinite IA case (elasticity goes to zero). Interestingly, this latter case corresponds to the maximin (d'Autume and Schubert, 2008a), which has resonance with assessing sustainability (Cairns and Martinet, 2014; Fleurbaey, 2015a).

I show that when society has to make an intergenerational sacrifice, three elements guide its sharing. The marginal productivity gap, the intragenerational inequality aversion and the intergenerational inequality aversion. The former indicates that the region having access to a relatively more productive asset (North) may afford a relative higher consumption. Paradoxically, the region having access to the relatively less productive asset (South) has to make a

[5] He considered the following properties: the utility is ordinal, continuous, satisfies sensitivity – one can distinguish two trajectories even if they differ only in their first element, – intertemporal complementarity – consumption on a given period does not impact the comparison between two alternatives on another period, – stationarity – comparison of two alternatives does not change when one goes forward in time – and if a best path and a worst path exist.

higher sacrifice. This comes from the fact that the South has a higher marginal productivity. The impact of the productivity gap on the consumption growth rates of the region is weighted by the two inequality aversions (IAs). The role played by the IAs can be ambiguous according to their level. But generally, the more one is willing to substitute welfare of each region, the more one can take advantage of their differences, then the sacrifices are more more unequal. At the opposite, when society exhibits an infinite intragenerational inequality aversion, growth rates of North and South are equal, whatever the productivity gap. And if society exhibits an infinite intergenerational inequality aversion, growth rates of North and South are equal (at the limit) to zero, since no sacrifice for the future is tolerated.

The next Section introduces the model: first in isolation, regions are collectively considered afterward. Section 3 exhibits links with the green accounting literature. I allow for an unequal treatment of regions and generations in Section 4. I discuss the relevance of discounting as a tool to prevent huge sacrifices in Section 5. Section 6 concludes.

2 Maximizing intergenerational welfare

I study an economy composed of two regions – North and South – with an infinite number of generations, each one consuming c_i of the asset $i = N,S$. I first study how each region in isolation can grow efficiently toward their highest sustainable level of utility, their golden rule (GR) u_i^{GR}. Then I consider a global social planner that derives an instantaneous social welfare function (SWF) from the utilities of the regions. And I study how this global economy can grow efficiently toward the global golden rule W^{GR}.

2.1 Framework

Each region is endowed with a stock of renewable resource X_i. The natural renewals F_i is assumed to be increasing, strictly concave and such that $F_i(0) = 0 = F_i(\bar{X}_i)$, where $\bar{X}_i$ stands for the carrying capacity.[6] They reach a maximum at the golden-rule stock X_i^{GR}. The dynamics of the stock is then given by

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} \equiv \dot{X}_i(t) = F_i(X_i(t)) - c_i(t) \ , \quad i = N,S \ . \tag{1}$$

Two approaches are possible to handle the welfare part. (1) Considering a constant elasticity of substitution (CES) SWF of consumptions (or utilities) from North and from South. And taking an increasing transformation of such a SWF to obtain a constant *intertemporal* elasticity of substitutions (CIES) SWF. (2) Considering a CIES utility function in each region with the

[6] I recognize that the dynamics of some resources cannot be represented by such functions. This is a stylized feature.

same elasticity parameter and representing them through a CES SWF. I represent them formally below.

Let θ be the intratemporal elasticity of substitution, with $\theta > 0$, $\theta \neq 1$. And let σ be the intertemporal elasticity of substitution, $\sigma > 0$, $\sigma \neq 1$. For the sake of simplicity, let us introduce the parameters $\eta \equiv \frac{\theta-1}{\theta}$ and $\nu \equiv \frac{\sigma-1}{\sigma}$, with $\eta, \nu < 1$ and $\eta, \nu \neq 0$. Those elasticities (θ and σ) can be interpreted as parameters of inequality aversion (Atkinson, 1970): from 0 (infinite inequality aversion) to infinity (nil inequality aversion). Let me also put regional weights $a_N, a_S > 0$ such that $a_N + a_S = 1$.[7] But for now consider only the case $a_N = a_S = \frac{1}{2}$. In mathematical terms, the two versions of the SWF are[8]

$$W^{(1)}(c_N, c_S) = \frac{\left(\left(a_N \cdot c_N^{\eta} + a_S \cdot c_S^{\eta}\right)^{\frac{1}{\eta}}\right)^{\nu}}{\nu}; \tag{2}$$

$$W^{(2)}(c_N, c_S) = \left(a_N \left(\frac{c_N^{\nu}}{\nu}\right)^{\eta} + a_S \left(\frac{c_S^{\nu}}{\nu}\right)^{\eta}\right)^{\frac{1}{\eta}}. \tag{3}$$

The first version allows to disentangle intertemporal from intratemporal distributional issues. Indeed the intertemporal substitution applies on each instantaneous aggregated SWF.[9] On the opposite, the second version allows to compare intertemporal decisions for each region and for the whole economy. With $\eta \to 1$ ($\theta \to \infty$) the global problem tends to be the sum of each regional ones. For this reason, I will follow the second approach (actually, it can easily be shown that the two approaches are formally equivalent[10]). Regional utilities are strictly concave but identical: $u_i(c_i) = \frac{c_i^{\nu}}{\nu}$. Here again, this captures the idea of responsibility for turning consumption into utility.

Let me now solve the intertemporal problem for our two regions considered separately. Afterward they will be considered collectively to allow for comparisons.

[7]The condition $a_N + a_S = 1$ (resp. the -1 in its intertemporal counterpart) is there only for having the Cobb-Douglass (resp. the logarithm) as a special case in the limit.

[8]To simply calculus, I omit the -1 usually put in CIES functions.

[9]See d'Autume and Schubert (2008a) and d'Autume et al. (2010) for models adopting this approach.

[10]Consider the following variable change: $\eta' = \nu\eta$. It comes $W^{(2)}(c_N, c_S) = \left(a_N \left(\frac{c_N^{\nu}}{\nu}\right)^{\frac{\eta'}{\nu}} + a_S \left(\frac{c_S^{\nu}}{\nu}\right)^{\frac{\eta'}{\nu}}\right)^{\frac{\nu}{\eta'}} = \frac{\left(a_N \cdot c_N^{\eta'} + a_S \cdot c_S^{\eta'}\right)^{\frac{\nu}{\eta'}}}{\nu}$.

2.2 Two regions considered separately

At the beginning, the regions are endowed with the stocks $X_i(0) = X_{i0}$. It is useful to compute the maximin consumption for each region in isolation. One directly obtains it by

$$c_i^m = \begin{cases} F_i(X_{i0}) & \text{if} \quad X_{i0} \leq X_i^{GR} \,; \\ F_i\left(X_{i0}^{GR}\right) & \text{if} \quad X_{i0} > X_i^{GR} \,. \end{cases} \tag{4}$$

If a region is endowed with a low stock (lower than the golden-rule stock), it can at best consume its regional production. Growth toward a higher consumption can come only at the price of a sacrifice, $c_i < F_i(X_i)$, over any period of time. On the contrary, if a region is endowed with an abundant stock (higher than the golden-rule stock), the GR $u_i^{GR} = u_i\left(F_i\left(X_i^{GR}\right)\right)$, is attainable at no cost. Indeed a path with dissavings, $c_i > F_i(X_i)$, is possible as long as the stock, when decreasing, does not overshoot the golden-rule stock. I am particularly interested here in the first scenario because reaching the GR calls for sharing sacrifice between generations. To do so, let us maximize the welfare of all future generations. To get a solvable problem, I resort to the classic Ramsey's (1928) device. Recall $u_i(c_i) = \frac{c_i^\nu}{\nu}$. The regional intertemporal welfares are given by[11]

$$v(X_{i0}) = \max_{c_i} \quad \int_0^\infty \left(u_i(c_i) - u_i^{GR}\right) \mathrm{d}t \,; \tag{5}$$

$$\text{subject to} \quad \dot{X}_i = F_i(X_i) - c_i \,; \tag{6}$$

$$X_{i0} \text{ given} \,. \tag{7}$$

I assume the utilities evolve fast enough such that the integrals are finite (discussion on convergence may be found in Chakravarty, 1962, and Chiang, 1992, pp. 99-101). The Hamiltonians are

$$\mathscr{H}_i(X_i, c_i, \psi_i) = u_i(c_i) - u_i^{GR} + \psi_i(F_i(X_i) - c_i) \,. \tag{8}$$

The necessary conditions are

$$\frac{\partial \mathscr{H}_i(X_i, c_i, \psi_i)}{\partial c_i} = 0 \quad \Leftrightarrow \quad \psi_i = c_i^{\nu-1} \,; \tag{9}$$

$$-\frac{\partial \mathscr{H}_i(X_i, c_i, \psi_i)}{\partial X_i} = \dot{\psi}_i \quad \Leftrightarrow \quad -\frac{\dot{\psi}_i}{\psi_i} = F_i'(X_i) \,; \tag{10}$$

$$\lim_{t\to\infty} \mathscr{H}_i(X_i, c_i, \psi_i) = 0 \,. \tag{11}$$

As usual, the shadow-price of a stock equals its marginal value in terms of utility (eq. (9)). Here, as there is no time preference, it always pays to save ($\dot{\psi}_i < 0 \Leftrightarrow \dot{c}_i > 0$) as long as the

[11] As a by-product of this analysis, the 'abundance' case – when the initial stock is higher than the golden-rule stock – is also handled: the objective maximize the current utility, net of its lower limit.

marginal return on capital is positive (eq. (10)). From the equations (9) and (10), one obtains

$$(1-\nu)\frac{\dot{c}_i}{c_i} = F_i'(X_i) \quad \Leftrightarrow \quad \frac{\dot{c}_i}{c_i} = \sigma F_i'(X_i) \; . \tag{12}$$

This is the so-called 'Keynes-Ramsey rule' without discount factor. The consumption growth rates depend positively on the marginal productivity, since it pays to save for growth, and positively on the intertemporal elasticity of substitution (IES), since a higher value of this parameter represents a higher willingness to substitute current for future welfares. Society saves more today in order to have a higher utility in the future. At the limit, if the parameter tends to infinity, the sacrifice at the benefit of future generations is maximal. At the other limit, the growth rate becomes constant, i.e. the sacrifice tends to be nil, when the IES approaches zero. No substitutions between current and future welfares are tolerated and one approaches the individual maximin consumptions c_i^m.[12]

As a preliminary for comparison with the collective problem, let me simply remark that the eq. (12) hold for North and South. One can therefore write

$$\frac{\dot{c}_S}{c_S} - \frac{\dot{c}_N}{c_N} = \sigma \left(F_S'(X_S) - F_N'(X_N) \right) \; . \tag{13}$$

The difference in regional growth rates equals the (marginal) productivity gap weighted by the IES.

With the transversality condition, the unique individual steady states are characterized by (the proof of the stability is omitted since this case is very classic)

$$c_i^\star = F_i\left(X_i^{GR}\right) \; , \; F_S'\left(X_S^{GR}\right) = 0 = F_N'\left(X_N^{GR}\right) \; , \; u_i(c_i^\star) = u_i^{GR} \; , \; i = N,S \; . \tag{14}$$

Integrating both sides of the equation (13) and using the final consumptions, it comes[13]

$$\frac{c_N(t)}{c_S(t)} = \frac{c_N^{GR}}{c_S^{GR}} \mathrm{e}^{\sigma \int_t^\infty \left(F_S'(X_{Sr}) - F_N'(X_{Nr})\right) \mathrm{d}r} \; . \tag{15}$$

I will come back to this equation for the collective problem.

[12] For a proof of a CIES approaching the maximin at a zero IES, see d'Autume and Schubert (2008a). See also d'Autume et al. (2010) for the undiscounted case.

[13] $\int_t^\infty \left(\frac{\dot{c}_S}{c_S} - \frac{\dot{c}_N}{c_N}\right) \mathrm{d}r = \int_t^\infty \sigma \left(F_S'(X_{Sr}) - F_N'(X_{Nr})\right) \mathrm{d}r + a \;\Leftrightarrow\; \left[-\ln\left(\frac{c_N}{c_S}\right)\right]_t^\infty = \sigma \int_t^\infty \left(F_S'(X_{Sr}) - F_N'(X_{Nr})\right) \mathrm{d}r + a \;\Leftrightarrow\; \ln\left(\frac{c_N(t)}{c_S(t)}\right) = \ln\left(\frac{c_N^{GR}}{c_S^{GR}}\right) + \sigma \int_t^\infty \left(F_S'(X_{Sr}) - F_N'(X_{Nr})\right) \mathrm{d}r.$

2.3 Two regions considered collectively

I now consider the problem of a 'world' social planner. At the beginning, the economy is still endowed with stocks (X_{N0}, X_{S0}). The maximin solution of this problem can be found in Cairns et al. (2016). A useful reference path is the *as-good-as-golden locus* (Phelps and Riley, 1978) obtained with the maximin framework: all stocks from which the welfare level is exactly that of the GR. It generalizes the GR concept to several dimensions. As in the previous subsection, I will mainly be interested in initial stocks lower than such a locus (this corresponds to the *regular part* of Cairns et al. (2016)). The GR is now given by $W^{GR} = W\left(c_N^{GR}, c_S^{GR}\right)$, with $c_i^{GR} = F_i\left(X_i^{GR}\right)$. Recall $W(c_N, c_S) = \left(a_N \left(\frac{c_N^{\nu}}{\nu}\right)^{\eta} + a_S \left(\frac{c_S^{\nu}}{\nu}\right)^{\eta}\right)^{\frac{1}{\eta}}$. Let us compute the global intertemporal welfare in the same manner than previously[14]

$$V(X_{N0}, X_{S0}) = \max_{c_N, c_S} \int_0^{\infty} \left(W(c_N, c_S) - W^{GR}\right) \mathrm{d}t \; ; \tag{16}$$

$$\text{subject to} \quad \dot{X}_N = F_N(X_N) - c_N \; ; \tag{17}$$

$$\dot{X}_S = F_S(X_S) - c_S \; ; \tag{18}$$

$$(X_{N0}, X_{S0}) \text{ given} \; . \tag{19}$$

Here also, I assume that the welfare evolves fast enough such that the integral is finite. The Hamiltonian is

$$\mathscr{H}(X_i, c_i, \psi_i) = W(c_N, c_S) - W^{GR} + \psi_N(F_N(X_N) - c_N) + \psi_S(F_S(X_S) - c_S) \; . \tag{20}$$

The necessary conditions are

$$\frac{\partial \mathscr{H}(X_i, c_i, \psi_i)}{\partial c_i} = 0 \quad \Leftrightarrow \quad \psi_i = a_i W^{1-\eta} \cdot \frac{c_i^{\eta\nu - 1}}{\nu^{\eta - 1}} \; ; \tag{21}$$

$$-\frac{\partial \mathscr{H}(X_i, c_i, \psi_i)}{\partial X_i} = \dot{\psi}_i \quad \Leftrightarrow \quad -\frac{\dot{\psi}_i}{\psi_i} = F_i'(X_i) \; ; \tag{22}$$

$$\lim_{t \to \infty} \mathscr{H}(X_i, c_i, \psi_i) = 0 \; . \tag{23}$$

From the equations (21) and (22), it comes

$$-(1-\eta)\frac{\dot{W}}{W} + (1-\eta\nu)\frac{\dot{c}_i}{c_i} = F_i'(X_i) \quad \Leftrightarrow \quad \frac{\dot{c}_i}{c_i} = \frac{\theta\sigma}{\theta + \sigma - 1}\left(F_i'(X_i) + \frac{1}{\theta}\frac{\dot{W}}{W}\right) \; . \tag{24}$$

The (apparent) consumption growth rates still depend on marginal productivity, but also on elasticities of substitutions in a subtle way. Let me first study the consumption growth rates when each elasticity reaches its limits. There are four cases.[15] Note $\theta \equiv \frac{1}{1-\eta}$ and $\sigma \equiv \frac{1}{1-\nu}$.

[14] Here also, as a by-product of this analysis, the 'abundance' case is also handled.

[15] The names of theses four cases are a misuse of language since the general case approaches the one mentioned.

L1. Intragenerational utilitarianism: $\theta \to \infty$, $\frac{\dot{c}_i}{c_i} \to \sigma F_i'(X_i)$.

L2. Intragenerational maximin: $\theta \to 0$, $\frac{\dot{c}_i}{c_i} \to \frac{\sigma}{\sigma-1}\frac{\dot{W}}{W}$.

L3. Intergenerational utilitarianism: $\sigma \to \infty$, $\frac{\dot{c}_i}{c_i} \to \theta F_i'(X_i) + \frac{\dot{W}}{W}$.

L4. Intergenerational maximin: $\sigma \to 0$, $\frac{\dot{c}_i}{c_i} \to 0$.

When the intragenerational IA is nil (case L1), the collective problem corresponds to the problem of two separated regions (see the eq. (12)). And only the intergenerational IA matters. In the same way, when the intergeneration IA is nil (L3), only the intragenerational IA matters. Interestingly, when the IA becomes infinite in one dimension (L2 or L4), the consumption growth rates do not longer depend on productivities, and are therefore equal. Intuitively, as no 'substitution of well-being' is tolerated with the maximin, there is no growth rates inequality. In the intragenerational maximin (L2), growth rates depend equally on the intergenerational dimension. When the intergenerational IA is infinite (L4), growth rates approach zero, and the sacrifice is very low. For the opposite reason, when IA become nil in the two dimensions ($\theta, \sigma \to \infty$), growth rates becomes infinite and then 'close'. I now turn the study of the difference in growth rates.

Let us equalize the common term in the eq. (24), for $i = N, S$, to obtain

$$\frac{\dot{c}_S}{c_S} - \frac{\dot{c}_N}{c_N} = \kappa\left(F_S'(X_S) - F_N'(X_N)\right) \ , \quad \text{with } \kappa \equiv \frac{\theta\sigma}{\theta+\sigma-1} \, . \tag{25}$$

Relatives growth of individual consumptions depend on the difference of marginal productivities, weighted by the intra and the intergenerational inequality aversions. Without loss of generality, I will only consider the case of a marginally more productive stock in the South, $F_S'(X_S) \geq F_N'(X_N)$, to capture the idea of a relative less abundant stock.

It may firstly be noted that the two elasticities have a different impact on the regional consumption paths (eq. (24)), but exactly the same on the difference of consumption growth rates (eq. (25)). Let me differentiate κ in order to study the relationship between the two dimensions. One has

$$-\frac{\mathrm{d}\sigma}{\mathrm{d}\theta}\bigg|_{\kappa} > 0 \quad \Leftrightarrow \quad \frac{1-\sigma}{1-\theta} > 0 \, . \tag{26}$$

Then, as far as the evolution of the difference in consumptions is concerned, the two IA are 'substitutes' if they are both sufficiently high ($\theta, \sigma < 1$) or both sufficiently low ($\theta, \sigma > 1$). And 'complements' in all other cases.

To better understand these features, I now study the sign of κ and make simple comparative statics.

- $\kappa > 0 \;\Leftrightarrow\; \theta + \sigma > 1$;

- $\frac{d\kappa}{d\theta} > 0 \Leftrightarrow \sigma > 1$ and $\frac{d\kappa}{d\sigma} > 0 \Leftrightarrow \theta > 1$.

Five cases have to be distinguished, as it is shown in Fig. 1.

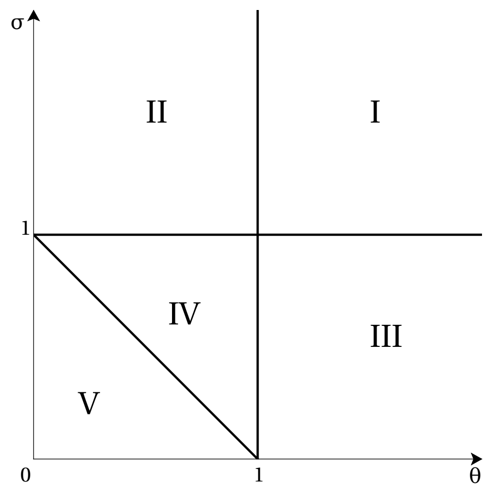

Figure 1: Possibilities for inequality aversions

Let us sum up the signs of level and variations of κ in each situation.

I. $\kappa > 0$; $\frac{d\kappa}{d\theta} > 0$; $\frac{d\kappa}{d\sigma} > 0$.

II. $\kappa > 0$; $\frac{d\kappa}{d\theta} > 0$; $\frac{d\kappa}{d\sigma} < 0$.

III. $\kappa > 0$; $\frac{d\kappa}{d\theta} < 0$; $\frac{d\kappa}{d\sigma} > 0$.

IV. $\kappa > 0$; $\frac{d\kappa}{d\theta} < 0$; $\frac{d\kappa}{d\sigma} < 0$.

V. $\kappa < 0$; $\frac{d\kappa}{d\theta} < 0$; $\frac{d\kappa}{d\sigma} < 0$.

Regarding sign, if intra and intergenerational IAs are low (I to IV), the difference in regional consumption rates is positive (see the eq. (25)). That means that everything else being equal, the higher the marginal productivity of the stock available in the South, the higher its consumption growth rate. Since the South has access to a more productive stock, a reduction of its consumption leads to a better return. South has therefore to make a relative higher sacrifice. This could seem surprising: taking into account intragenerational concerns in a classic accumulation model leads to rise the difference of optimal development between unequal regions. Rather, it seems rational to take advantage of initial differences, but of course this calls for redistribution mechanism both for ethical and for practical reasons of implementing such a decision rule. Interestingly, if intra and intergenerational IAs are sufficiently high (V), the previous reasoning is reverse and North has to make a relative higher sacrifice. In any case, when marginal productivities converge toward each other (approaching the GR), the regional consumption growth rates converge too. The levels of consumption converge as well, as explained below.

Regarding variations, if intra and intergenerational IAs are sufficiently low (I), the lower the IA, the higher the difference in regional consumption growth rates (see the eq. (25)). On the opposite, and counter-intuitively, if both IA are high (IV and V), the lower the IA, the lower the difference in regional consumption growth rates. There are two intermediary cases when the IA is low in one dimension but high in the other dimension (II and III). They correspond to the 'complementarity' cases displayed in the eq. (26).

Let me now have a word on the comparison between regions in isolation and regions collectively considered. Of course, regional consumption growth rates differ in the two problem (compare eq. (12) and (24)). But as far as the difference in growth rates in concerned, the problem reduces in comparing σ (eq. (13)) with κ (eq. (25)). It is not hard to see that $\kappa > \sigma$ if and only if $\sigma < 1$ and $\theta + \sigma > 1$. In other words, the collective problem lead to a higher difference in consumption growth rates than the individual problem only in cases III and IV. They correspond to a case where both the intragenerational IA is relatively low (intuitively, a low IA lead to high differences) and the intergenerational IA is high (counter-intuitively). Let me now go back to the study of the limits, but for κ only.

L1. Intragenerational utilitarianism: $\theta \to \infty$, $\kappa \to \sigma$: if the intragenerational IA is nil, the difference of consumption growth rates depends only on the intertemporal IA and on the productivity gap (as in the problem of two separated regions).

L2. Intragenerational maximin: $\theta \to 0$, $\kappa \to 0$: if the intragenerational IA is infinite, the regional consumptions grow at the same rate.

L3. Intergenerational utilitarianism: $\sigma \to \infty$, $\kappa \to \theta$: if the intergenerational IA is nil, the difference of consumption growth rates depends only on the intratemporal IA and on the productivity gap.

L4. Intergenerational maximin: $\sigma \to 0$, $\kappa \to 0$: if the intergenerational IA is infinite, the regional consumptions grow at the same rate.

L1 and L3 corresponds to cases when the 'distribution of well-being' does not count in one dimension. Then the relative sacrifice depends only on the IA of the other dimension. Cases L2 and L4 lead to the same result of an equal evolution of the consumption growth rates, but for different reasons. In the case L2, the infinite intragenerational IA leads the growth rates to be as close as possible. While in the case L4, one refrains oneself as much as possible to substitute a current (lower) consumption for a future (higher) consumption. This leads the growth rates to be as small as possible, and therefore to be close. In other words, if the IA is infinite in at least one dimension consumptions evolve at the same rate. If the IA is nil in one dimension, the difference in growth rate depends only on the IA of the other dimension. And finally, if the IAs are nil in the two dimensions, the social welfare is nearly only supported by the consumption of the North (low sacrifice) while the South let its stock grow as fast as possible (huge sacrifice). Consumptions approach a corner solution.[16] In this last case, the productivity gap is fully exploited. It calls even more clearly for a transfer from the North to the South.

Finally, it shall be noted that in case of equal marginal productivities (e.g. if natural renewal are identical), the IAs play no role and the consumptions grow at the same rate toward their golden-rule levels.

With the transversality condition, the unique steady state is characterized by (the stability is shown in the Appendix A)

$$c_i^{\star\star} = F_i\left(X_i^{GR}\right)\ ,\ F_S'\left(X_S^{GR}\right) = 0 = F_N'\left(X_N^{GR}\right)\ ,\ W(c_N^{\star\star}, c_S^{\star\star}) = W^{GR}\ . \tag{27}$$

Integrating both sides of the equation (25) and using the final consumptions, it comes (as for the eq. (15))

$$\frac{c_N(t)}{c_S(t)} = \frac{c_N^{GR}}{c_S^{GR}} \mathrm{e}^{\kappa \int_t^\infty \left(F_S'(X_{Sr}) - F_N'(X_{Nr})\right)\mathrm{d}r}\ . \tag{28}$$

This equation helps understanding the evolution of the consumptions through their ratio. Let me concentrate on cases where $\kappa > 0$ (i.e. $\theta + \sigma > 1$). As long as the stock of the South is more productive than that of the North, the current ratio is higher than the final ratio and converges toward it. It shows, more directly, how a more productive stock leads to a higher sacrifice. But to visualize this feature even more easily, I turn to a graphical representation.

2.4 Graphical representation

Fig. 2 is a four-quadrant plot that represents the different elements. The upper-right quadrant plots the evolution of the consumption and the stock of the North. Symmetrically, the lower-left plots these elements for the South. In such a way, the upper-left quadrant plots the consumption paths. And the lower-right quadrant plots the stock paths.

The upper-right and lower-left quadrants are, individually, similar to a one-sector Ramsey model.[17] They depict a saddle point: consumption has to increase (resp. decrease) if the initial stock is lower (resp. higher) than the golden-rule stock. For the sake of the representation, I use the transversality condition (reaching the GR) to project the $\dot{c}_i = 0$ curves in the upper-left quadrant and the $\dot{X}_i = 0$ curves in the lower-right one. The upper-left and lower-right quadrants represent then only optimal paths, ending up at the steady state M (corresponding to M'). The

[16]This limit is easily computable with the notation $\kappa \equiv \frac{1}{1-\eta\nu}$. When $\eta, \nu \to 1,\ \kappa \to \infty$.

[17]It is direct from the eq. (21) and (22), noticing that $\psi_i > 0$ is a decreasing function of c_i: $\frac{\partial \psi_i}{\partial c_i} = \frac{\partial^2 W(\cdot)}{(\partial c_i)^2} < 0$.

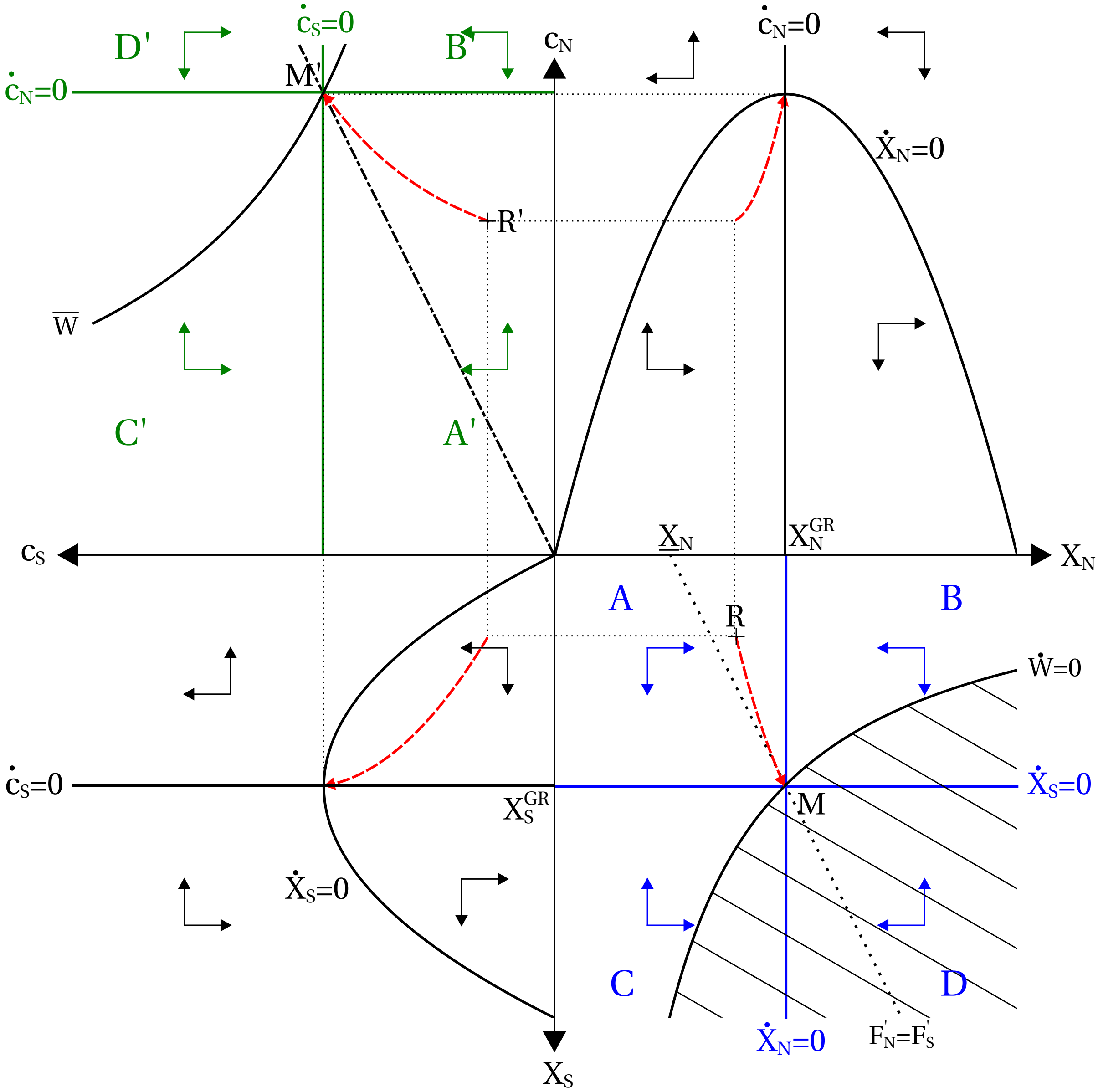

Figure 2: Global evolutions of the consumptions and the stocks

paths never cross the $F_N' = F_S'$ locus[18] ($\underline{X}_N M$ and $0M'$), otherwise one would have a steady state that does not satisfy the transversality condition.[19] I conjecture that paths never cross the curve passing through M (M') depicting $\dot{W} = 0$ ($W = W^{GR}$).[20] As mentioned before, this curve represents the as-good-as-golden locus. The hatched area depicts stocks from which the social welfare decreases toward its steady-state value. More generally, no cycle occurs (in consumptions or states) since paths cannot cross the 'critical loci'.

[18] It is a straight line because I plotted symmetric production functions.

[19] Approaching the steady state (possibly asymptotically), stock and consumption paths have the same tangent (by L'Hospital's rule): $\lim_{t\to\infty} \frac{\mathrm{d}X_N}{\mathrm{d}X_S} = \lim_{t\to\infty} \frac{F_N'(X_N)\dot{X}_N - \dot{c}_N}{F_S'(X_S)\dot{X}_S - \dot{c}_S} = \frac{\mathrm{d}c_N}{\mathrm{d}c_S} = \frac{c_N^{GR}}{c_S^{GR}}$.

[20] A discussion on this point can be found in Samuelson and Solow (1956).

An example of path is plotted from R' to M'. It is very similar to a classic path in a Ramsey model. But we can now better visualize the impact of κ on the consumption ratio, exposed in the eq. (28). The lower κ, the closer the current ratio to the final ratio. The consumptions path gets closer and closer to the straight line $0M'$. For example, if the IAs are initially low ($\theta, \sigma > 1$), the higher the inequality aversions, the lower the actual inequalities during the convergence. But the previous explanations showed that this intuitive feature does not always hold.

Finally, let me have a word on the 'regular' part (the non-hatched area) of the area B. From initial stocks in this part, the North has an abundant stock $X_{N0} > X_N^{GR}$ while the South has a scarce stock $X_{S0} < X_S^{GR}$. Here North shall overshoot its production to decrease toward the GR, while South shall make classic savings to grow. Clearly, here, we could take a part of the 'manna' in North to compensate the sacrifice in South. But how regional sacrifices contribute to the improvement of the welfare? This is depicted by Green accounting tools.

3 Accounting considerations

During the transition toward the GR, savings are made. One expects therefore an indicator of sustainability to be increasing. What I interpreted as 'sacrifice' at any given date is given by the difference between the production $F_i(X_i)$ and the current consumption c_i, i.e. by the distance between the current and the sustainable consumption, for $X_i < X_i^{GR}$. But this difference was already measured by the evolution of the stocks since $\dot{X}_i = F_i(X_i(t)) - c_i(t)$. In other words, the instantaneous sacrifice made by a region is represented by $\dot{X}_i$ and its total instantaneous sacrifice by $X_i^{GR} - X_{i0}$. And, as well-known, the marginal impact of a stock on the value function V is given by shadow-prices ψ_i. I study this formally.

In autonomous problems, the optimal Hamiltonian is constant (see e.g. Chiang, 1992, p. 190). The transversality condition (23) implies therefore $\mathscr{H}^{\text{opt}} = 0$. The equation (20) can then be rewritten as

$$\underbrace{\psi_N \dot{X}_N + \psi_S \dot{X}_S}_{\text{Net investment}} = \underbrace{W^{GR} - W}_{\text{Welfare gap}} . \tag{29}$$

Along a rising path, $W < W^{GR}$, net investment (NI) is positive and is linked to the 'welfare distance'. The farther the economy is from the target, the more it has to invest. Each region makes a contribution of $\psi_i \dot{X}_i$. And investment tends toward zero as W approaches W^{GR}. Once at the GR is reached (possibly asymptotically) one obtains, in a sense, the Hartwick's rule (Hartwick, 1977; Dixit et al., 1980; Withagen and Asheim, 1998).[21] Actually, this feature was already found by Ramsey (1928) himself – it is the 'genuine' Keynes-Ramsey rule – and was restated with exhaustible resource and capital accumulation by d'Autume and Schubert (2008b). Here, positive, nil and negative investments could be possible according to the initial states.

[21]This always holds for the as-good-as-golden locus.

Let me now turn to the final analysis when a unequal treatment is considered.

4 Intra and intergenerational inequity

From a purely positivist view, one may consider treating both regions and generations unequally. Actually, Ramsey (1928) himself considered discounting at the end of his paper, while he strongly rejected it on an ethical ground at the beginning. In the same vein, Koopmans (1965) used a discount rate to compare discounted and undiscounted versions. At any time, the discount rate may be nil to come back to the previous case of undiscounted welfare.

Let me follow this approach by considering discounting as well. What represents a higher weight on current and near welfares than futures ones. What Heal (2009, p. 5) called as "*the rate of intergenerational discrimination*". By the same token, let me also attach different weights on regions to see if changes happen. These intra and intergenerational weights may be interpreted as a measure of (procedural) 'inequity'.

Let a_N and a_N be now elements of $(0,1)$, still with $a_N + a_S = 1$. And let us introduce a positive discount rate δ. I study only the collective problem, but at any time, recall that letting $\theta \to \infty$ allows to approach individual ones.

The intertemporal welfare is now given by

$$V^d(X_{N0}, X_{S0}) = \max_{c_N, c_S} \quad \int_0^\infty \mathrm{e}^{-\delta t} \left(W(c_N, c_S) - W^{GR} \right) \mathrm{d}t \; ; \tag{30}$$
$$\text{subject to the same constraints .}$$

The eq. (24) becomes

$$\frac{\dot{c}_i}{c_i} = \frac{\theta\sigma}{\theta+\sigma-1} \left(F_i'(X_i) + \frac{1}{\theta}\frac{\dot{W}}{W} - \delta \right) \; . \tag{31}$$

In comparison with the situation without discounting, consumption growth rates are lower. Intuitively, if future generations count less, we less need to save in order to increase their welfares. And less investments lead to a lower growth. Apparently, intragenerational weights do not play a role here. Indeed, they impact W but they have the same impact on the evolution of the consumptions (same $\frac{\dot{W}}{W}$ for both). Actually, they impact the levels of optimal consumptions. To see this let us rewrite the green accounting equation.

$$\psi_N \dot{X}_N + \psi_S \dot{X}_S = \mathrm{e}^{-\delta t} \left(W^{GR} - W \right) \; . \tag{32}$$

As ψ_i depends positively on a_i, the 'sacrifice sharing' depends well on those weights. Everything else being equal, a higher weight a_i means a lower $\dot{X}_i$, then a lower sacrifice.

Interestingly, when the IA is infinite in at least one dimension (L2 or L4), intergenerational weights (discounting) disappear in the eq. (31).[22] They become "immaterial" (d'Autume and Schubert, 2008b). This feature is well-known in a static framework: when the elasticity of substitution tends to zero, a CES function tends to the *symmetric* minimum. This common feature both in intra and in intergenerational dimensions is not so surprising. Weights (attached to individuals or generations) guide the priority of resources. But, in parallel, the elasticity of substitution tells to what extend society is willing to substitute the well-being of an individual (or generation) for that of another one. When the elasticity tends to zero, substitution are no longer tolerated. Therefore, whatever the attached weights, the absolute priority is given to the worst-off. Only in this case, weights do not matter. This is even clearer when $\theta \to \infty$ ($\eta \to 1$):

$$\frac{\dot{c}_i}{c_i} = \sigma \left(F_i'(X_i) - \delta \right) \ . \tag{33}$$

In the classic so-called 'Keynes-Ramsey rule', a nil IES 'erases' the discount rate. Let us now merge the two equations (31), for $i = N, S$. One still has

$$\frac{\dot{c}_S}{c_S} - \frac{\dot{c}_N}{c_N} = \kappa \left(F_S'(X_S) - F_N'(X_N) \right) \ . \tag{34}$$

Therefore, the evolution of the inequality between the consumption growth rate in the North and in the South is robust to both intra and intergenerational 'inequity'. But as said before, the levels of consumption are impacted. In particular, the steady state is now characterized by a lower welfare, the modified golden rule. It is reached when

$$F_S'(X_S) = F_N'(X_N) = \delta \ . \tag{35}$$

That means that the sacrifice shall stop when the marginal productivities are both equal to the pure rate of time preference. Taken in another way, the discount rate can be chosen so as to meet any states on the $F_S' = F_N'$ locus, and then limits the global sacrifice. This brings me to my last discussion.

5 Discussion: Equity, inequality aversion and discount rate

The Ramsey criterion has been criticized for leading to a significant sacrifice of every generation at the benefit of subsequent ones (Arrow, 1999). As argued by Rawls (1971, p. 287): "*the utilitarian doctrine may direct us to demand heavy sacrifices of the poorer generations for the sake of greater advantages for later ones that are far better off. But this calculus of*

[22] When $\theta \to 0$ discounting disappear but intragenerational weights do still play a (undifferentiated) role on consumption growth rates through W (L2). But when $\sigma \to 0$, no weight matter (L4).

advantages, which balances the losses of some against benefits to others, appears even less justified in the case of generations than among contemporaries". But as Rawls (1971, sec. 44) refrained himself to apply either discounting or the difference principle criterion ('maximin') in an intertemporal perspective, he gave no precise recommendations about the optimal savings rate.[23] And he admitted that "*it is not possible, at present anyway, to define precise limits on what the savings rate should be*" (p. 286). To find a solution, some authors prefer to adopt a consequentialist approach and to discuss the resulting allocation of well-being of a particular criterion. For example, Zuber and Asheim (2012) allow for weights according to the rank of generational well-being. This was built up on the "Hammond Equity for the Future" axiom (Asheim and Tungodden, 2004), which captures the following idea: a sacrifice of the present for the future is (weakly) desirable if the present stays better-off than the future. In this view, discounting comes from the expression of inequality aversion if future generation are better-off (Asheim and Mitra, 2010; Asheim, 2012; Zuber and Asheim, 2012; Asheim and Ekeland, 2016). I see several potential problems with the view of 'equitable discounting'.

The intratemporal part allows to better understand the distinct role of weights and of the intragenerational IA. Undesirable consequences from unequal weights can indeed be reduced with a higher inequality aversion. And when IA is infinite, weights disappear. But why changing weights rather than the IA? In the present framework, sacrifices may also be reduced by two ways: with a higher intertemporal IA or with a discount rate. The first option (consider $\theta > 1$ for simplicity) leads to make regional consumption paths closer when growing toward the GR. The second option leads to a situation where sacrifice is less needed. And then each generation has to make a lower sacrifice. Both options may be desirable, but in light of this framework it seems odd to justify discounting by the IA. By the way, such a practice would be intolerable in the intragenerational dimension. Also, I find odd that the IA be dependent upon the situation. How to justify that such a change in the IA arises merely because the situation changed? I think that linking directly the IA with the specific situation can lead to arbitrary decisions. The underlying justification that future can benefit from current sacrifice, more than proportionally, does not seem to justify *per se* a departure from impartiality (Fleurbaey and Michel, 1999, p. 723). Besides, if equity requires that something is equalized, I do not see precisely what is equalized with discounting. That said, resulting allocation of well-being may be worth discussing, but one needs, I think, another vocable.

Giving the priority to the worst-off individuals because they are worse-off is advocated by proponents of *prioritarianism* (Parfit, 1995; Fleurbaey, 2015b). In this view, weights should

[23] It is sometimes mentioned that Rawls (1971) acknowledged the interest of discounting to avoid sacrifices. But one shall add that he was in contradiction with the primitive problem. He wrote: "*introducing time preference may be an improvement in such cases; but I believe that its being invoked in this way is an indication that we have started from an incorrect conception. [...] It is introduced in a purely ad hoc way to moderate the consequences of the utility criterion*" (p. 298).

be inversely related to the relative initial well-being. Here, in the intertemporal dimension, it would justify discounting for increasing welfare paths. In the intratemporal dimension, it would also recommend to give more weight to the South as long as it is less well-off than the North.

I propose to argue that the undiscounted utilitarian criterion is sufficiently malleable to handle different intergenerational inequality outcomes (Fleurbaey and Michel, 1994, 1999; Asheim and Buchholz, 2007). Which comes back to asking, ultimately, what is the appropriate shape of the welfare function (Schelling, 1995). But this malleability comes at a price when intratemporal issues are taken into account. Even if the difference in relative sacrifices reduces over time, the inequality may be important during the transition path. Giving the same weight to the regions turned out to be insufficient.

6 Conclusion

This simple framework enables us to have straightforward insights. When regions have access to unequally productive renewable resources, the sacrifice to attain the highest sustainable level of welfare – the golden rule – is not evenly shared. Basically, the region which is endowed with a lower stock, South, has to make a higher sacrifice than the other one, North. The intuition behind this startling result is that the resource of the South, being lower than that of the North, is marginally more productive. The same sacrifice leads then to a higher return in the South than in the North. The difference in regional growth rates depends generally positively on the productivity gap, and negatively on the intratemporal and the intertemporal inequality aversions. The sacrifice for reaching the golden rule is evenly shared *only* in three cases. (1) If the marginal productivities are equal, because one cannot take advantage of differences of the two regions. (2) If the intratemporal inequality aversion is infinite, because one does not want to take advantage of differences of the two regions. (3) If the intertemporal inequality aversion is infinite, because one does not want to make sacrifice. The (original) Keynes-Ramsey rule holds in this framework: the farther the economy is from the golden rule, the more it has to invest. At the end, unequal weights put on regions and on generations were considered. I came to two conclusions. (i) Discounting cannot be advocated on the sole basis of the willingness to limit sacrifices. (ii) Intratemporal weights do not play a fundamental role in the relative regional growth rates.

The main conclusion of this model, based on the most efficient way to share sacrifice in order to grow toward the golden rule, can be difficult to hear in the real world. Both from an ethical and a pragmatic view, such a policy can only be implemented with transfers from the North to the South.

A Stability of the steady state

Let me sum up the necessary conditions into three main equations to get the following dynamic system, with $Y \equiv \ln\left(\frac{c_S}{c_N}\right)$;

$$\begin{cases} \dot{X}_N &= F_N(X_N) - c_N \;; \\ \dot{X}_S &= F_S(X_S) - c_S \;; \\ \dot{Y} &= \kappa\left(F'_S(X_S) - F'_N(X_N)\right) \;. \end{cases} \tag{36}$$

Steady states are characterized by (using the transversality condition)

$$\begin{cases} c_N^{GR} &= F_N\left(X_N^{GR}\right) \;; \\ c_S^{GR} &= F_S\left(X_S^{GR}\right) \;; \\ F'_N\left(X_N^{GR}\right) &= F'_S\left(X_S^{GR}\right) \;. \end{cases} \tag{37}$$

Consider the Jacobian matrix of the linearized system, evaluated at the steady states[24]

$$J^\star = \begin{pmatrix} F'_N & 0 & \alpha_N \\ 0 & F'_N & -\alpha_S \\ -\kappa F''_N & \kappa F''_S & 0 \end{pmatrix} \;; \tag{38}$$

with $\alpha_N \equiv -\frac{\partial c_N}{\partial Y}\Big|_{c_N^{GR}, c_S^{GR}} = \frac{\left(c_S^{GR}\right)^2}{c_N^{GR}} > 0$ and $\alpha_S \equiv \frac{\partial c_S}{\partial Y}\Big|_{c_N^{GR}, c_S^{GR}} = c_S^{GR} > 0$.

Let us compute the roots of the characteristic polynomial $\mathscr{P}(\lambda) = \det\left(J^{GR} - \lambda I_3\right)$:

$$\left(F'_N - \lambda\right)\left(-\left(F'_N - \lambda\right)\lambda + \kappa\alpha_N F''_N + \kappa\alpha_S F''_S\right) = 0\,. \tag{39}$$

The first eigenvalue is $\lambda_N = F'_N$. Let $\Gamma \equiv -\kappa\left(\alpha_N F''_N + \alpha_S F''_S\right) > 0$. I can reduce the eq. (39) to $\lambda^2 - F'_N\lambda - \Gamma = 0$. Eigenvalues are then $\lambda_N = F'_N > 0$, $\lambda_S = \frac{F'_N - \sqrt{(F'_N)^2 + 4\Gamma}}{2} < 0$, and $\lambda_3 = \frac{F'_N + \sqrt{(F'_N)^2 + 4\Gamma}}{2} > 0$.[25] The steady state is a (classic) saddle-point.

[24] We use the equality $F'_N = F'_S$ and express the Jacobian with respect to F'_N only.

[25] Notice that strict concavity of production functions rule out nil eigenvalues.

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