Social Acceptability of Condorcet Committees

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Abstract

We define and examine the concept of social acceptability of committees, in multi-winner elections context. We say that a committee is socially acceptable if each member in this committee is socially acceptable, i.e., the number of voters who rank her in their top half of the candidates is at least as large as the number of voters who rank her in the least preferred half, otherwise she is unacceptable. We focus on the social acceptability of Condorcet committees, where each committee member beats every non-member by a majority, and we show that a Condorcet committee may be completely unacceptable, i.e., all its members are unacceptable. However, if the preferences of the voters are single-peaked or single-caved and the committee size is not "too large" then a Condorcet committee must be socially acceptable, but if the preferences are single-crossing or group-separable, then a Condorcet committee may be socially acceptable but may not. Furthermore, we evaluate the probability for a Condorcet committee, when it exists, to be socially (un)acceptable under Impartial Anonymous Culture (IAC) assumption. It turns to be that, in general, Condorcet committees are significantly exposed to social unacceptability.

Keywords: Voting, Multiwinner Elections, Committee, Condorcet, Social Acceptability.

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1. Introduction

In multi-winner elections, the goal is to select a subset of candidates (i.e., a committee) of a pre-given size, such as electing parliaments, shortlisting job candidates or choosing public locations for a set of facilities, such as hospitals or fire stations (Faliszewski et al., 2017). In this paper we

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generalize and examine the concept of social acceptability of candidates, which has been introduced by Mahajne and Volij (2018a) for single-winner elections context, to multi-winner election setting.

Consider a set of candidates. We say that a voter places a given candidate *above the line* if she prefers her to at least half of the candidates, and she places her *below the line* if at least half of the candidates are preferred to her.[3] We say that a candidate is *socially acceptable* with respect to a given preference profile, if she is placed above the line by at least as many voters as those who place her below the line. To be socially unacceptable, may be a significant weakness for a candidate, because this means that more voters place this candidate in their least favorite half of the candidates rather than in their most favorite half.

Social acceptability of committees is defined in this paper as follows. Given a preference profile, we say that a committee is *socially acceptable* if all its members are socially acceptable, and it is *completely unacceptable* if all its members are socially unacceptable and it is *partly unacceptable*[4] if some of its members are unacceptable. In multi-winner elections, to be a socially unacceptable committee may be a significant disadvantage, because this means that all or some of the committee members are socially unacceptable, as shown for instance in Example 1 later. A socially unacceptable committee may cause dissatisfaction in some sense and a majority of voters may feel disappointed or even skeptical regarding some or all its elected members.

This paper, which extends the work of Mahajne and Volij (2018b) regarding single-winner elections, examines the social acceptability of Condorcet committees. There are several definitions of *a Condorcet committee* as generalizations of the concept of Condorcet winner. One of them, which we consider in this paper, is due to Gehrlein (1985). A Condorcet committee *à la* Gehrlein is a committee such that every one of its members beats every non-member by a majority (Aziz et al., 2017; Coelho, 2004; Elkind et al., 2015; Kamwa, 2017).

We show that in general, a Condorcet committee may be partly and even completely unacceptable. However, if the preferences of the voters are single-peaked or single-caved and the committee size does not exceed a certain number, then a Condorcet committee must be socially acceptable, and if the preferences are single-crossing or group-separable, then it may be socially acceptable but may not. Single-crossingness, which can guarantee Condorcet winner to be socially acceptable (Mahajne

[3] Concerning the reason why exactly half of the candidates are considered, a discussion can be found in Mahajne and Volij (2018a).

[4] This also means that it is *partly acceptable* since some of its members are acceptable. However we use the term *partly unacceptable* because we see the situation in which some of the committee members are socially unacceptable as negative, by analogy to the Condorcet committee principle.

and Volij, 2018b), cannot guarantee Condorcet committee (of size 2 or more) to be socially acceptable.

These previous types of preferences may be not likely to exist; therefore we also evaluate the probability of a Condorcet committee to be socially unacceptable under the commonly used assumption of Impartial Anonymous Culture (IAC). This assumption, introduced by Gehrlein and Fishburn (1976), is a commonly used hypothesis in the literature of social choice theory when computing the theoretical likelihood of electoral events, and stipulates that all voting situations (defined later) are equally likely to be observed. Our results show that in general, under IAC, Condorcet committees are exposed to social unacceptability to a significant extent. For instance, when there are six candidates, then about 80 percent of the voting situations lead to a Condorcet committee of size four are expected to be partly or completely unacceptable.

The paper is organized as follows. Section 2 lays out the basic definitions.[5] Section 3 states and proves theoretical results regarding social acceptability of Condorcet committees. Section 4 presents the results regarding the probability evaluations of having a socially (un)acceptable Condorcet committee under IAC assumption, and Section 5 concludes.

2. Definitions

Let $A = \{a_1, a_2, \dots, a_K\}$ be a set of $K \geq 3$ candidates and let $N = \{1, \dots, n\}$ be a set of $n \geq 3$ voters. We consider a framework where each voter is assumed to have a linear order $\succ$ on A, from the most desirable candidate to the least desirable one. Let $\mathcal{P}$ be the set of all linear orders on A. We refer to the elements of $\mathcal{P}$as preference relations. We denote by π the preference profile of the voters, it is summarized by a list $\pi = (\succ_1, \dots, \succ_n)$ of n preference relations, where for each voter $i \in N$, $\succ_i$ represents i's preference relation over the candidates in A. We denote by $\mathcal{P}^n$ the set of preference profiles. Let π be a preference profile, for any subset $C \subseteq \mathcal{P}$ of preference relations, $\mu_\pi(C) = |\{i \in N : \succ_i \in C\}|$ is the number of voters whose preferences are in C. Also, $\pi(N) = \{\succ \in P : \exists i \in N \ s.t. \succ_i = \succ\}$ denotes the set of different preference relations that are present in the profile π. For any preference relation $\succ \in \mathcal{P}$, the inverse of $\succ$ is the preference relation $\succ^{-1}$ defined by $a \succ^{-1} b \Leftrightarrow b \succ a$. Let $\pi = (\succ_1, \dots, \succ_n)$ be a preference profile, the *inverse profile* of π is the profile $\pi^{-1} = (\succ_1^{-1}, \dots, \succ_n^{-1})$. Let $a, a' \in A$ be two candidates. Denote by $C(a \succ a') = \{\succ \in \mathcal{P} : a \succ a'\}$ the set of preference relations according to which a is preferred to a'. Along this paper, when we write for instance $\succ = (b_1 b_2 \dots b_K)$, we mean that b_1 is placed first in $\succ$ and b_2 is placed

[5] Mainly, we use similar basic definitions as in Mahajne and Volij (2018b).

second and so on. For any preference relation $\succ$ and for any candidate $a \in A$, the rank of a in $\succ$, is defined as: $rank_{\succ}(a) = K - |\{a' \in A: a \succ a'\}|$. Candidates whose ranks in $\succ$ are less than $(K + 1)/2$ are said to be placed *above the line* by $\succ$ and candidates whose ranks are greater than $(K + 1)/2$ are said to be placed *below the line* by $\succ$, and candidates whose ranks are equal to $(K + 1)/2$ are said to be placed *on the line* by $\succ$. For instance, if $K = 4$ and a voter's preference relation is given by $a_1 \succ a_2 \succ a_3 \succ a_4$, then she places candidates a_1 and a_2 above the line and candidates a_3 and a_4 below the line. In this example, no candidate is placed on the line because the number of candidates is even.

We are interested in multi-winner elections where we need to elect a fixed size subset of candidates (a committee) from the given set of candidates A. Let k $(k < K)$ denote the committee size or cardinality, and let $C(A) = \{\mathbb{C}_1, \dots, \mathbb{C}_M\}$ be the set of all k-size committees of A. A *multi-winner voting rule* is a function that assigns to each preference profile π, a k-size subset of A.

The concept of *social acceptability* has been introduced by Mahajne and Volij (2018a), for a single-winner elections context:

Definition 1 Let π be a profile of preference relations, and let $a \in A$ be a candidate. We say that a is *socially acceptable* with respect to π if the number of voters that place her above the line is at least as large as the number of voters that place her below the line and otherwise a is *socially unacceptable.* Formally, *a is socially acceptable* with respect to π if and only if

$$\mu_\pi(\{\succ: rank_{\succ}(a) < (K + 1)/2\}) \geq \mu_\pi(\{\succ: rank_{\succ}(a) > (K + 1)/2\})$$

We extend the concept of social acceptability to multi-winner context and present a definition of *social acceptability of committees* as follows:

Definition 2 Let π be a preference profile, and let $\mathbb{C} \in C(A)$ be a committee. We say that $\mathbb{C}$ is *socially acceptable* with respect to π if every committee member $a \in \mathbb{C}$ is socially acceptable with respect to π. We say that $\mathbb{C}$ is *socially partly unacceptable* if some of its members are socially unacceptable and some are socially acceptable. We say that $\mathbb{C}$ is *socially completely unacceptable* if all its members are socially unacceptable.[6]

[6] Henceforth, we sometimes may use in short "*partly unacceptable*" and "*completely unacceptable*", without "*socially*".

3. Social acceptability of Condorcet committees

3.1 General

We now provide the definition of *Condorcet committee à la Gehrlein*, and later we examine the social acceptability of Condorcet committees, especially under some common preference restrictions such as single-peakedness. Before, we present a definition of *Condorcet winner*:

Definition 3 Let π be a preference profile, and let $a \in A$ be a candidate. We say that a is a *(strong) Condorcet winner* with respect to π if and only if:[7]

$$\mu_\pi\big(C(a \succ a')\big) > \mu_\pi\big(C(a' \succ a)\big) \;\; \forall a' \in A\backslash\{a\}$$

The definition of Condorcet winner has been extended to multi-winner context in different ways, one of them is due to Gehrlein (1985): A committee $\mathbb{C} \in C(A)$ is a *Condorcet committee à la Gehrlein* if each member in this committee beats each non-member by a majority. Formally,

Definition 4 Let π be a preference profile, and let $\mathbb{C} \in C(A)$ be a committee. We say that $\mathbb{C}$ is a *Condorcet committee à la Gehrlein* with respect to π if and only if:

$$\mu_\pi(C(a \succ a')) > \mu_\pi(C(a' \succ a)) \;\; \forall a \in \mathbb{C} \; and \; \forall a' \in A\backslash\mathbb{C}$$

The Condorcet committee *à la* Gehrlein does not always exist, and as pointed by Diss et al. (2018), it has been suggested in order to avoid committees with dominated members. For more information on *Condorcet* committee, the reader can refer in particular to Barberà and Coelho (2008), Kamwa (2017), and Ratliff (2003).

The first Proposition of this paper shows that when there are 3 candidates, every Condorcet committee of size 1 (i.e., a Condorcet winner) must be socially acceptable, and every Condorcet committee of size 2 can be socially acceptable or partly unacceptable but it cannot be completely unacceptable. However, when there are more than 3 candidates, all the situations are possible.

[7] If we replace "$>$" by "$\geq$", then we say that it is (*weak*) *Condorcet winner*, and when we write *"Condorcet winner"* we mean (*strong*) *Condorcet winner*, unless we write "*weak*".

Proposition 1 *Let $A = \{a_1, \ldots, a_K\}$, let π be a preference profile, and let $\mathbb{C} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π.*

a) If $K = 3$ and $k = 1$, then $\mathbb{C}$ must be socially acceptable.

b) If $K = 3$ and $k = 2$, then $\mathbb{C}$ can be socially acceptable or partly unacceptable but it cannot be completely unacceptable.

c) If $K > 3$, then all the situations are possible.

Proof: Denote: $\succ^1 = (abc), \succ^2 = (acb), \succ^3 = (bac), \succ^4 = (cab), \succ^5 = (bca), \succ^6 = (cba)$, and denote ($\forall j = 1, \ldots, 6$): $n^j = |\{i \in N : \succ_i = \succ^j\}|$. We prove now the **first statement**. Let $k = 1$ and let a (w.l.g) be a (strong) Condorcet winner (it must exists, otherwise $\mathbb{C}$ will be empty). In this case, $\mathbb{C} = \{a\}$, thus we obtain that: $\mu_\pi(C(a \succ a')) \geq \mu_\pi(C(a' \succ a))\ \ \forall a' \in A \backslash \{a\}$.[8] Therefore,

$$n^1 + n^2 + n^3 \geq n^4 + n^5 + n^6 \text{ (since } a \text{ beats } c \text{ by a majority)}$$

$$n^1 + n^2 + n^4 \geq n^3 + n^5 + n^6 \text{ (since } a \text{ beats } b \text{ by a majority)}$$

Thus, $2n^1 + 2n^2 + n^3 + n^4 \geq n^3 + n^4 + 2n^5 + 2n^6$. Consequently, $n^1 + n^2 \geq n^5 + n^6$. But:

$$n^1 + n^2 = \mu_\pi(\{\succ : rank_\succ(a) = 1\}) = \mu_\pi(\{\succ : rank_\succ(a) < (K+1)/2\})$$

$$n^5 + n^6 = \mu_\pi(\{\succ : rank_\succ(a) = 3\}) = \mu_\pi(\{\succ : rank_\succ(a) > (K+1)/2\})$$

Therefore, $\mu_\pi(\{\succ : rank_\succ(a) < (K+1)/2\}) \geq \mu_\pi(\{\succ : rank_\succ(a) > (K+1)/2\})$. That is, a is socially acceptable, and thus $\mathbb{C}$ is socially acceptable.

The **second statement**. Since $k = 2$, we consider (w.l.g) that $\mathbb{C} = \{a, b\}$. In this case, one of them (a w.l.g) must be a (weak) Condorcet winner, i.e., $\mu_\pi(C(a \succ a')) \geq \mu_\pi(C(a' \succ a))\ \ \forall a' \in A \backslash \{a\}$ (otherwise, $\mathbb{C}$ will be empty). But, by the proof of the first statement, a must be socially acceptable. Therefore, at least one member of $\mathbb{C}$ is socially acceptable. The next example shows that for $K = 3$ and $k = 2$, $\mathbb{C}$ can be socially acceptable or partly unacceptable.

Example 1 Let $A = \{a, b, c\}$ *and consider the following two profiles:* $\pi_1 = \{(abc), (bac), (cba)\}$ *and* $\pi_2 = \{(abc), (abc), (acb)\}$. *It can be seen that under π_1, $\{a, b\}$ is a Condorcet committee and it is socially acceptable* (one *voter place a above the line and one below the line*, one *voter place b above the line and* no one *below the line*), *and that under π_2, $\{a, b\}$ is a Condorcet committee but it is partly unacceptable (a is socially acceptable, b is socially unacceptable*).

[8] It holds '>', but '≥' is useful for statement 2.

The next example completes the proof (the **third statement**) and shows that for $K > 3$, all the situations are possible, i.e., a Condorcet committee may be socially acceptable or partly unacceptable or completely unacceptable.

Example 2 *Let* $A = \{a, b, c, d\}$ *and consider the following four profiles:*
$\pi_1 = \{(acbd), (acdb), (cabd)\}$, $\pi_2 = \{(abcd), (acbd), (cdab), (cbad), (bdac)\}$,
$\pi_3 = \{(abcd), (cdab), (cbad), (bdac), (cbad), (cabd), (adcb), (abcd), (bdca)\}$,
$\pi_4 = \{(abcd), (cdab), (cbad), (dbac), (cbad), (cadb), (adcb), (abcd), (dbca)\}$
It can be checked that under π_1, $\{a, c\}$ *is a Condorcet committee that is socially acceptable (*3 *voters place each one of them above the line but no one below the line), and under* π_2, $\{a, c\}$ *is a Condorcet committee that is partly unacceptable (*c *is acceptable but* a *is unacceptable), and under* π_3, $\{a, c\}$ *is a Condorcet committee that is completely unacceptable, and that also under* π_4, $\{a, c, d\}$ *is a Condorcet committee that is completely unacceptable.* ■

The implications of selecting unacceptable committees may be significantly negative on the satisfaction of the voters. In Example 1, under π_3 there is a majority of voters (5 out of 9) that may be "unsatisfied" regarding each member of the Condorcet committee $\{a, c\}$, which is completely unacceptable. If for instance, the following function defines the satisfaction of voter i from candidate a,

$$S_{\succ_i}(a) = \begin{Bmatrix} 1 & if\ rank_{\succ_i}(a) < (K+1)/2 \\ 0 & if\ rank_{\succ_i}(a) = (K+1)/2 \\ -1 & if\ rank_{\succ_i}(a) > (K+1)/2 \end{Bmatrix}$$

and the total satisfaction of the voters from committee $\mathbb{C}$ is $S_\pi(\mathbb{C}) = \sum_{i=1}^{n} S_{\succ_i}(\mathbb{C})$ (where $S_{\succ_i}(\mathbb{C}) = \sum_{a \in \mathbb{C}} s_{\succ_i}(a)$*)*, then we obtain in Example 1, that under π_3, $\{b, d\}$, which is socially acceptable, is a committee which yields best total satisfaction level of 2, whereas the Condorcet committee $\{a, c\}$ yields negative total satisfaction level of -2.

After the previous analysis of the relationship between social acceptability and Condorcet committees, now we will focus on the relation between *q-Condorcet committee* and social acceptability. We define and propose the q-Condorcet committee principle as a generalization of the well-known *q-Condorcet winner* to the multiwinner setting. We follow the definition of Baharad and Nitzan (2003) and Courtin et al. (2015a,b) who define the q-Condorcet winner as a candidate who is never beaten by another candidate with a fraction *q* of the number of voters.

Definition 5 Let π be a preference profile, and let $\mathbb{C} \subset A$ be a committee of size k. For $q \in (\frac{1}{2}, 1)$, we say that $\mathbb{C}$ is a $q - Condorcet\ committee$ (*à la* Gehrlein) with respect to π if for every candidate $a \in \mathbb{C}$ and every candidate $a' \in A \backslash \mathbb{C}$ the number of voters that prefer a to a' is greater than a fraction q of the number of voters. Namely, if

$$\mu_\pi\big(C(a \succ a')\big) > qn, \textit{ for all } a \in \mathbb{C}, a' \in A\backslash\mathbb{C}.$$

The next proposition shows that when q is large enough, any $q - Condorcet\ committee$ cannot be completely unacceptable or even must be socially acceptable if q is too large.

Proposition 2 Let $q_0 = \frac{3K-2k-2}{4(K-k)}$ and $q_1 = \frac{3K-2k-2}{4k(K-k)} + \frac{k-1}{k}$ where $q_0 < q_1$, $q_0 < 1$, and $q_1 < 1$. Let π be a preference profile for which $\mathbb{C}$ is a *q-Condorcet committee.*

a) Let $q \geq q_0$, then $\mathbb{C}$ cannot be completely unacceptable.

b) Let $q \geq q_1$, then $\mathbb{C}$ is socially acceptable.

Proof :[9] **Statement a**. Let $\mathbb{C} = \{c_1, \dots, c_k\}$ be a *q-Condorcet committee* with respect to π and let

$$W_\pi(\mathbb{C}) = \sum_{a\in\mathbb{C}} W_\pi(a) = \sum_{a\in\mathbb{C}} \underbrace{\sum_{a'\in A\backslash\mathbb{C}} \mu_\pi(C(a \succ a'))}_{W_\pi(a)}$$

Since $\mu_\pi\left(C\left(a \succ a'\right)\right) > q_0.n \geq \frac{3K-2k-2}{4(K-k)}n$ *for all* $a \in \mathbb{C}, a' \in A\backslash\mathbb{C}$, we have that $W(\mathbb{C}) > nq_0.k(K-k) \geq nk(K-k)\frac{3K-2k-2}{4(K-k)}$. Therefore, we obtain that

$$W(\mathbb{C}) > nk\left(\frac{3K-2k-2}{4}\right) \qquad (1)$$

Assume by contradiction that $\mathbb{C}$ is completely unacceptable. There are two cases:

Case 1: K is even. Then, $\forall c_j \in \mathbb{C}$ there is a proportion $a_j < 1/2$ of voters that place c_j above the line and a proportion $1 - a_j$ of voters that place c_j below the line. Let π' be the preference profile that is obtained from π by sending the candidates in $\mathbb{C}$ to the *top places* (in the *upper half* of A) of each preference relation that places them above the line, and by sending the candidates in $\mathbb{C}$ to the top places of the *lower half* of A (i.e., in the top places of $\{K/2 + 1, \dots, K\}$ starting just below the

[9] Following the proof of Proposition 1 in Mahajne and Volij (2018b).

line) of each preference relation that places them below the line. By construction, $\mathbb{C}$ is still completely unacceptable and $W_\pi(c_j) \le W_{\pi'}(c_j)$ *for all* $c_j \in \mathbb{C}$.

When sending c_j to the top places of the *upper half*, she obtains $K-k$ "wins" on non-member candidates (i.e., she is preferred to $K-k$ non-member candidates) and when sending c_j to the top places of the *lower half*, she obtains at most $(K/2-1)$ "wins" on non-member candidates. Therefore *for all* $c_j \in \mathbb{C}$,

$$W_\pi(c_j) \le W_{\pi'}(c_j) \ \le na_j(K-k) + n(1-a_j)(\tfrac{K}{2}-1) = n[a_j(\tfrac{K-2k+2}{2}) + \tfrac{K-2}{2}]$$

Since $a_j < 1/2$, and since $(K-2k+2) > 0$,[10] we have that:

$$W_\pi(c_j) < n\left[\left(\frac{K-2k+2}{4}\right) + \frac{K-2}{2}\right] = n(\frac{3K-2k-2}{4})$$

Therefore, we have that:

$$W_\pi(\mathbb{C}) = \sum_{c_j \in \mathbb{C}} W_\pi(c_j) < \sum_{c_j \in \mathbb{C}} n(\frac{3K-2k-2}{4}) \ = nk(\frac{3K-2k-2}{4})$$

This contradicts inequality 1.

Case 2: K is odd. $\forall c_j \in \mathbb{C}$, there is a proportion $a_j < \ 1/2$ of voters that place c_j above the line and a proportion β_j of voters that place c_j below the line ($a_j < \ \beta_j$). Let π' be the preference profile that is obtained from π by sending the candidates in $\mathbb{C}$ to the $top\ places$ (in the *"upper half"* of A) of each preference relation that places them above the line, and by sending the candidates in $\mathbb{C}$ to the top places of the *lower half* of A (i.e., in the top places of $\{(K+1)/2\ +\ 1, \dots, K\}$, starting just below the line) of each preference relation that places them below the line. By construction, $\mathbb{C}$ is still completely unacceptable, and by a previous argument we have $\forall c_j \in \mathbb{C}$

$$W_\pi(c_j) \le W_{\pi'}(c_j) = na_j(K-k) + n\beta_j\left(\frac{K-1}{2} - 1\right) + n(1-a_1-\beta_j)(\frac{K-1}{2})$$

$$= na_j(K-k) + n(1-a_j)\left(\frac{K-1}{2}\right) - n\beta_j$$

Since $a_j < \ \beta_j$, we have $W_\pi(c_j) < na_j(K-k) + n(1-a_j)\left(\frac{K-1}{2}\right) - na_j = n[a_j\,\frac{(K-2k-1)}{2} + \frac{K-1}{2}]$

Since $a_j < \ 1/2$, and since $(K-2k-1) > 0$ we have $\forall c_j \in \mathbb{C}$

$$W_\pi(c_j) < n\frac{1}{2}[\frac{(K-2k-1)}{2} + \frac{K-1}{2}] = n\frac{3K-2k-3}{4}$$

Therefore, $W_\pi(\mathbb{C}) = \sum_{c_j \in \mathbb{C}} W_\pi(c_j) < \sum_{c_j \in \mathbb{C}} n[\frac{(3K-2k-2)}{4}] = nk[\frac{3K-2k-2}{4}]$, which contradicts inequality 1.

[10] $q_0 = \frac{3K-2k-2}{4(K-k)} < 1 \Leftrightarrow k < \frac{K+2}{2}$, but then $(K-2k+2) > 0$.

Statement b. Since $\mu_\pi\left(C\left(a \succ a'\right)\right) > \ q_1.n \geq (\frac{3K-2k-2}{4k(K-k)} + \frac{k-1}{k})n$ *for all* $a \in \mathbb{C}\,, a' \in A\backslash\mathbb{C}$, we have that $W(\mathbb{C}) > \ nq_1.k(K-k) \geq nk(K-k)(\frac{3K-2k-2}{4k(K-k)} + \frac{k-1}{k})$. Therefore, we obtain that

$$W(\mathbb{C}) > \ n\left(\frac{3K-2k-2}{4} + (k-1)(K-k)\right) \qquad (2)$$

Assume by contradiction that $\mathbb{C}$ is (at least) partly unacceptable. There are two cases:

Case 1: K is even. Then, $\forall c_j \in \mathbb{C}$ there is a proportion $a_j \leq 1$ of voters that place c_j above the line and a proportion 1 – a_j of voters that place c_j below the line. Let π' be the preference profile that is obtained from π by sending the candidates in $\mathbb{C}$ to the $top\ places$ (in the *upper half* of A) of each preference relation that places them above the line, and by sending the candidates in $\mathbb{C}$ to the top places of the *lower half* of A (i.e., in the top places of $\{K/2\ +\ 1,\ldots,K\}$, starting just below the line) of each preference relation that places them below the line. By construction, $\mathbb{C}$ is still partly unacceptable and $W_\pi(c_j) \leq W_{\pi'}(c_j)$, $\forall c_j \in \mathbb{C}$. By a previous argument we have $\forall c_j \in \mathbb{C}$

$$W_\pi(c_j) \leq W_{\pi'}(c_j) \ \leq na_j(K-k) + n\left(1-a_j\right)(\tfrac{K}{2}-1) = n[a_j(\tfrac{K-2k+2}{2}) + \tfrac{K-2}{2}]$$

Since for at least one c_j (assume w.l.g $c_j = c_1$) $a_j < 1/2$, and since $(K-2k+2) > 0,$[11] we have that: $W_\pi(c_1) < n\left[\left(\frac{K-2k+2}{4}\right) + \frac{K-2}{2}\right] = n(\frac{3K-2k-2}{4})$. Since for $j > 1$, $a_j \leq 1$, we have that:

$$W_\pi\left(c_j\right) \leq n(\tfrac{K-2k+2+(K-2)}{2}) = n(K-k).$$

Therefore,

$$W_\pi(\mathbb{C}) = \sum_{c_j \in \mathbb{C}} W_\pi(c_j) < n\left(\frac{3K-2k-2}{4}\right) + (k-1)n(K-k)$$

$$= n[\left(\frac{3K-2k-2}{4}\right) + (k-1)(K-k)]$$

This contradicts inequality 2.

Case 2: K is odd. Then, $\forall c_j \in \mathbb{C}$ there is a proportion $a_j < \ 1/2$ of voters that place c_j above the line and a proportion β_j of voters that place c_j below the line. Let π' be the preference profile that is obtained from π by sending the candidates in $\mathbb{C}$ to the $top\ places$ (in the *"upper half"* of A) of each preference relation that places them above the line, and by sending the candidates in $\mathbb{C}$ to the top places of the *lower half* of A (i.e., in the top places of $\{(K+1)/2\ +\ 1,\ldots,K\}$, starting just below the line) of each preference relation that places them below the line. By construction, $\mathbb{C}$ is still not socially acceptable, and by a previous argument we have $\forall c_j \in \mathbb{C}$

[11] $q_1 = \frac{3K-2k-2}{4k(K-k)} + \frac{k-1}{k} < 1 \Leftrightarrow k < \frac{K+2}{2}$, but then $(K-2k+2) > 0$.

$$W_\pi(c_j) \le W_{\pi'}(c_j) = na_j(K-k) + n\beta_j\left(\frac{K-1}{2} - 1\right) + n(1 - a_1 - \beta_j)(\frac{K-1}{2})$$
$$= na_j(K-k) + n(1-a_j)\left(\frac{K-1}{2}\right) - n\beta_j$$

Since for at least one c_j (assume w.l.g $c_j = c_1$), $a_j < \beta_j$ and $a_j < 1/2$, and since $(K - 2k - 1) > 0$, we have $W_\pi(c_1) \le na_1(K-k) + n(1-a_1)\left(\frac{K-1}{2}\right) - na_1 = n[a_1 \frac{(K-2k-1)}{2} + \frac{K-1}{2}]$

$$< n\frac{1}{2}[\frac{(K-2k-1)}{2} + \frac{K-1}{2}] = n\frac{3K-2k-3}{4}$$

for $j > 1$, $W_\pi(c_j) \le na_j(K-k) + n(1-a_j)\left(\frac{K-1}{2}\right) - n\beta_j = n[a_j\left(\frac{K-2k+1}{2}\right) + \left(\frac{K-1}{2}\right) - \beta_j]$

Since for $j > 1$, $a_j \le 1$, $\beta_j \le 1$, we have that

$$W_\pi(c_j) \le n\left[\frac{K-2k+1}{2} + \left(\frac{K-3}{2}\right)\right] = n[(K-k) - 1] < n(K-k)$$

Therefore, $W_\pi(\mathbb{C}) = \sum_{c_j \in \mathbb{C}} W_\pi(c_j) < n\left[\frac{(3K-2k-2)}{4}\right] + (k-1).\, n(K-k)$

$$= n[\left(\frac{3K-2k-2}{4}\right) + (k-1)(K-k)]$$

This contradicts inequality 2.

The bounds q_0 and q_1 cannot be improved. To check this, let $K = 4$ and $k = 2$, so that the bounds are $q_0 = 3/4$ and $q_1 = 7/8$. We focus on q_0, so let $q < 3/4$. We will construct a preference profile for which the committee $\{a, b\}$ is a $q - Condorcet\ committee$ but is completely unacceptable. Let m be a positive integer such that $q < 9m/(12m + 1)$ and let π be a preference profile with: $2m$ voters with preference $(adbc)$, $2m$ voters with preference $(acbd)$, $2m$ voters with preference $(bcad)$, $2m$ voters with preference $(bdac)$, m voters with preference $(adbc)$, m voters with preference $(acbd)$, m voters with preference $(bcad)$, m voters with preference $(bdac)$, and 1 voter with preference $(cdab)$. The number of voters is $n = 12m + 1$. The number of voters who prefer each one of $\{a, b\}$ to each one of $\{c, d\}$ is $9m$. Therefore, $\{a, b\}$ is a $q - Condorcet\ committee$, but is completely unacceptable because each one of $\{a, b\}$ is placed above the line by $6m$ voters and below the line by $6m + 1$ voters.[12] ■

The main message of the previous results is that the Condorcet (and the q-Condorcet) committee and social acceptability are two notions which may be difficult to conjugate. As a result, in the next sections we focus on the social acceptability of Condorcet committees under some common domain conditions. We consider four cases which are extensively studied in the literature of social choice

[12] We can construct a "similar" preference profile for q_1 such that a committee is q-Condorcet committee but is partly unacceptable.

theory, namely, *single-peaked preferences*, *single-crossing preferences*, *single-caved preferences, and group-separable preferences.*

3.2 Condorcet committees under Single-peaked preferences

The class of single-peaked preferences, first introduced by Black (1948), is perhaps the most extensively studied type of domain restrictions. Roughly speaking, a set of preference relations are single-peaked with respect to a given linear order of the candidates if each preference has a peak such that for any two candidates on the same side of the peak, one is preferred over the other if she is closer to the peak. Formally,

Definition 6 Let $\leq$ be a linear order on the set of candidates A. We say that the preference relation $\succ$ is *single-peaked* with respect to $\leq$ if there is a candidate $p \in A$ (denoted also $p(\succ)$) such that

$$(a < b \leq p \text{ } or \text{ } p \leq b < a) \Rightarrow b \succ a$$

If $a < b$ we say that a is *to the left* of b or that b is *to the right* of a.We denote the set of all preferences that are single-peaked with respect to $\leq$ by $P(\leq)$. We say that the profile $\pi = (\succ_1, \ldots, \succ_n)$ is *single-peaked* with respect to a linear order $\leq$ on A, if all the preferences in $\pi(N)$ are single-peaked with respect to $\leq$. The next claim is useful,

Claim 1 *Let $\leq$ be a linear order on A and assume without loss of generality that $a_1 < \cdots < a_K$. Let $\succ$ be a single-peaked with respect to $\leq$. Let $a, b, c \in A$ any three candidates such that $a < c < b$, then it holds that: $(a \succ b) \Rightarrow (c \succ b)$ and $(b \succ a) \Rightarrow (c \succ a)$. And as a result we have that: $\mu_\pi(C(a \succ b) \leq \mu_\pi(C(c \succ b))$ and $\mu_\pi(C(b \succ a)) \leq \mu_\pi(C(c \succ a))$.*

Proof : The proof is presented in Appendix A.

Fix a linear order $\leq$ on A, the following lemma states first that for any candidate a who is not the “middle candidate” with respect to $\leq$, there is another candidate b (we call her the $counter\ of\ a$) such that for any single-peaked relation $\succ$ with respect to $\leq$, a is placed *above the line* if and only if a is preferred to b. Also it states that the number of candidates which are located, with respect to $\leq$, “between” any candidate a and *her counter* is at least $(K + 1)/2$.

Lemma 1 *Let $\leq$ be a linear order on $A = \{a_1, \ldots, a_K\}$ and assume without loss of generality that $a_1 <, \ldots, < a_K$. Then, for any $a \in A$ such that $a \neq a_{(K+1)/2}$:*

1) There is $b \in A$ (the counter of a) such that for all preferences $\succ$ that are single-peaked with respect to $\leq$, $rank_{\succ}(a) < \frac{K+1}{2} \Leftrightarrow a \succ b$; [13]

2) If we denote by $M(a)$ the counter of a, then it holds that:

- *if $a < M(a)$ then $|\{b \in A: a \leq b \leq M(a)\}| \geq (K+1)/2$*
- *if $M(a) < a$ then $|\{b \in A: M(a) \leq b \leq a\}| \geq (K+1)/2$.*

Proof : The proof is presented in Appendix C.

The next claim shows that under single-peaked profile, if $\mathbb{C}$ is a *Condorcet committee à la Gehrlein*, then for any two candidates in the committee, it must be that every candidate which is "between" them is also in the committee.

Claim 2 *Let $\leq$ be a linear order on A and assume without loss of generality that $a_1 < \ldots < a_K$. Let π be a preference profile of single-peaked preferences with respect to $\leq$, and let $\mathbb{C} = \{c_1, \ldots c_k\} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π. Then $\mathbb{C}$ is a continuous with respect to $\leq$, i.e., $\nexists c \in A$ such that: $a < c < b$ for some $a, b \in \mathbb{C}$ and $c \notin \mathbb{C}$.*

Proof : The proof is presented in Appendix B.

The next Corollary shows that under single-peaked profile, if $\mathbb{C}$ is a *Condorcet committee à la Gehrlein*, then for any candidate a in $\mathbb{C}$, its *counter* (denoted by $M(a)$) cannot be in the committee $\mathbb{C}$.

Corollary 1 *Let $\leq$ be a linear order on the of candidates A and assume without loss of generality that $a_1 < \ldots < a_K$. Let π be a single-peaked preference profile with respect to $\leq$, and let $\mathbb{C} = \{c_1, \ldots c_k\} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π and $|\mathbb{C}| < (K+1)/2$. Let a be any candidate such that $a \neq a_{(K+1)/2}$ and M(a) is the counter of a (according to Lemma 1). Then there cannot be the case that $a \in \mathbb{C}$ and $M(a) \in \mathbb{C}$.*

[13] As it is shown in the proof, if $a = a_i$ for some $i \leq \left\lceil \frac{K-1}{2} \right\rceil$, then $b = a_{\left\lceil \frac{K-1}{2} \right\rceil + i}$, and if $a = a_i$ for some $i \geq \left\lfloor \frac{K+1}{2} \right\rfloor + 1$, then $b = a_{i - \left\lceil \frac{K-1}{2} \right\rceil}$.

Proof: Assume by contradiction that $a \in \mathbb{C}$ and $M(a) \in \mathbb{C}$. By Claim 2, $\mathbb{C}$ is continuous, consequently we obtain that: **If** $\mathbf{a} < M(\mathbf{a})$ then $\{b \in A: a \leq b \leq M(a)\} \subseteq \mathbb{C}$, and thus $|\{b \in A: a \leq b \leq M(a)\}| \leq |\mathbb{C}|$, but by statement 2 of Lemma 1, $|\{b \in A: a \leq b \leq M(a)\}| \geq (K+1)/2$, thus $|\mathbb{C}| \geq (K+1)/2$, which contradict the assumption that $|\mathbb{C}| < (K+1)/2$. **If** $\boldsymbol{M(a)} < a$ then $\{b \in A: M(a) \leq b \leq a\} \subseteq \mathbb{C}$ and thus $|\{b \in A: M(a) \leq b \leq a\}| \leq |\mathbb{C}|$ but by statement 2 of Lemma 1, $|\{b \in A: M(a) \leq b \leq a\}| \geq (K+1)/2$, thus $|\mathbb{C}| \geq (K+1)/2$, which also contradict the assumption that $|\mathbb{C}| < (K+1)/2$. ■

The next Proposition shows that if the preferences are single-peaked with respect to some linear order on A and the committee size is not "too large", then a *Condorcet committee* must be socially acceptable. However, if the committee size exceeds half the number of the candidates, then a Condorcet committee may be socially acceptable or partly unacceptable, but it cannot be completely unacceptable.

Proposition 3 *Let* $\leq$ *be a linear order on A and assume without loss of generality that* $a_1 < \ldots < a_K$. *Let* π *be a single-peaked preference profile with respect to* $\leq$, *and let* $\mathbb{C} \in C(A)$ *be a Condorcet committee à la Gehrlein with respect to* π.

1) If $|\mathbb{C}| < (K+1)/2$, *then* $\mathbb{C}$ *is socially acceptable with respect to* π.

2) If $|\mathbb{C}| \geq (K+1)/2$, *then* $\mathbb{C}$ *can be socially acceptable with respect to* π *or partly unacceptable but it cannot be completely unacceptable.*

Proof: **Statement (1)**. Assume $|\mathbb{C}| < (K+1)/2$. First, if there is any $a \in \mathbb{C}$ such that $a \neq a_{(K+1)/2}$, then by Lemma 1 there is $b \in A$ (*it is* $M(a)$ *the counter* of a) such that for all $\succ \in P(\leq)$, $rank_{\succ}(a) < (K+1)/2 \Leftrightarrow a \succ b$. Therefore, $\mu_\pi(\{\succ: rank_{\succ}(a) < (K+1)/2\}) = \mu_\pi(C(a \succ b))$. Since $a \in \mathbb{C}$, then by Corollary 1, $b \notin \mathbb{C}$, and since $\mathbb{C}$ is a Condorcet committee, we must have that $\mu_\pi\big(C(a \succ b)\big) > \mu_\pi\big(C(b \succ a)\big)$, and so $\mu_\pi(C(a \succ b)) \geq n/2$. Therefore, $\mu_\pi(\{\succ: rank_{\succ}(a) < (K+1)/2\}) \geq n/2$. Consequently, $\mu_\pi(\{\succ: rank_{\succ}(a) < (K+1)/2\}) \geq \mu_\pi(\{\succ: rank_{\succ}(a) > (K+1)/2\})$. That is, a is socially acceptable with respect to π.

Now, if there is any $a \in \mathbb{C}$ such that $a = a_{(K+1)/2}$ (in this case K must be odd), then it must be that $rank_{\succ}(a) \leq (K+1)/2$ for all $\succ \in P(\leq)$, because otherwise (i.e., if $rank_{\succ}(a) > (K+1)/2$), and since $\succ$ is single-peaked with respect to $\leq$, we must have one of the two cases,

$$rank_{\succ}(d) > (K+1)/2, \forall d \in A \text{ such that } d \leq a \qquad (1)$$

$$rank_{\succ}(d) > (K+1)/2, \forall d \in A \text{ such that } a \leq d \qquad (2)$$

In case (1), there will be at least $i = (K+1)/2$ candidates with $rank_{\succ}(.) > (K+1)/2$, which is more than $(K-1)/2$, but this is impossible (it must be exactly $(K-1)/2$).

In case (2), there will be at least $K - i + 1 = K - (K+1)/2 + 1 = (K-1)/2 + 1$ candidates with $rank_{\succ}(.) > (K+1)/2$, which is more than $(K-1)/2$, but this is impossible.

Therefore, $\mu_{\pi}(\{\succ: rank_{\succ}(a) > (K+1)/2\}) = 0$, and consequently, $\mu_{\pi}(\{\succ: rank_{\succ}(a) < (K+1)/2\}) \geq \mu_{\pi}(\{\succ: rank_{\succ}(a) > (K+1)/2$. That is, a is socially acceptable. Thus, we proved that for any $a \in \mathbb{C}$ (regardless of her location with respect to $\leq$), a is socially acceptable. Consequently, $\mathbb{C}$ is socially acceptable with respect to π.

Statement 2. Assume $|\mathbb{C}| \geq (K+1)/2$. **First** we show that at least one Condorcet committee member must be socially acceptable. This results comes immediately from the proof of statement 1. Since π is single-peaked, we obtain that a (weak) Condorcet winner, which exists, must be in $\mathbb{C}$, because otherwise $\mathbb{C}$ will be empty. Let a be a (weak) Condorcet winner ($a \in \mathbb{C}$). According to the proof**,** if $a \neq a_{(K+1)/2}$, there is $b \in A$ such that for all $\succ \in P(\leq)$, $rank_{\succ}(a) < (K+1)/2 \Leftrightarrow a \succ b$. Consequently, $\mu_{\pi}(\{\succ: rank_{\succ}(a) < (K+1)/2\}) \geq \mu_{\pi}(\{\succ: rank_{\succ}(a) > (K+1)/2\})$. That is, a is socially acceptable. If $a = a_{(K+1)/2}$, then also by the proof, we obtain that a is socially acceptable. Therefore, $\mathbb{C}$ cannot be completely unacceptable. The next example completes the proof and shows that under single-peaked profiles, a Condorcet committee can be socially acceptable or partly unacceptable.

Example 3 *Let $A = \{a, b, c, d\}$ and consider the following two single-peaked profiles (with respect to the order $a < b < c < d$): $\pi_1 = \{(abcd), (bcad),\}$, $\pi_2 = \{(abcd), (abcd), (abcd)\}$. It can be checked that under π_1, $\mathbb{C} = \{a, b, c\}$ is a Condorcet committee and it is socially acceptable and that under π_2, $\mathbb{C} = \{a, b, c\}$ is a Condorcet committee but it is partly unacceptable (a, b are acceptable, c is unacceptable). Similar examples can be given for any K and $|\mathbb{C}| \geq (K+1)/2$.* ■

3.3 Condorcet committees under Single-caved preferences

The concept of single-caved preferences was introduced by Inada (1964, 1969). A set of preference relations are *single-caved* with respect to a given linear order of the candidates if each preference relation has a *"cave"* candidate such that for any two candidates on the same side of the "cave", one is preferred over the other if she is more distant from the "cave". On the other side, single-caved preference profiles are generated from single-peaked profiles by inversing the preference relation of each voter. As pointed by Barberà et al. (2012), single-caved preferences can arise, for instance, in the presence of an undesirable suggested project or policy, like construction of a prison. In this case, for a voter, the worst location may be the closest to his home, and as the location is further away it is more desirable. Formally,

Definition 7 Let $\leq$ be a linear order on the set of candidates A. We say that the preference relation $\succ$ is *single-caved* with respect to $\leq$ if there is a candidate $d \in A$ (denoted also, $cave(\succ)$) such that

$$(a < b \leq d \text{ } or \text{ } d \leq b < a) \Rightarrow a \succ b$$

If $a < b$ we say that a is *to the left* of b or that b is *to the right of* a. We denote the set of all preferences that are single-caved with respect to $\leq$ *by* $PC(\leq)$. We say that the profile π is *single-caved* with respect to a linear order $\leq$ on A, if all the preferences in $\pi(N)$ are single-caved with respect to $\leq$.

The next claim, regarding single-caved preferences, is useful:

Claim 3 *Let* $\leq$ *be a linear order* on the set of candidates A *and assume without loss of generality that* $a_1 < \cdots < a_K$*. Let* $\succ$ *be a single-caved with respect to* $\leq$*.*
Let $a, b, c \in A$ *any three candidates such that* $a < b < c$*, then it holds that:*
$(b \succ a) \Rightarrow (c \succ b)$ *and as a result:* $\mu_\pi(C(b \succ a) \leq \mu_\pi(C(c \succ b))$
$(b \succ c) \Rightarrow (a \succ b)$ *and as a result:* $\mu_\pi(C(b \succ c) \leq \mu_\pi(C(a \succ b))$

Proof: The proof is given in Appendix D.

The following lemma states that for any candidate a that is not "middle candidate" with respect to $\leq$, there is another candidate b (we call her the $counter\ of\ a$) such that for any single-caved relation $\succ$ with respect to $\leq$, a is above the line if and only if a is preferred to b.

Lemma 2 *Let* $\leq$ *be a linear order* on the set of candidates A *and assume without loss of generality that* $a_1 <\ldots, < a_K$*. Then, for any* $a \in A$ *such that* $a \neq a_{(K+1)/2}$ *:*
1) There is $b \in A$ *(the counter of* a*) such that for all preferences* $\succ$ *that are single-caved with respect to* $\leq$*,* $rank_\succ(a) < (K + 1)/2 \Leftrightarrow a \succ b$*;*[14]

[14] As it is shown in the proof, if $a = a_i$ for some $i \leq \left\lceil \frac{K-1}{2} \right\rceil$, then $b = a_{\left\lfloor \frac{K+1}{2} \right\rfloor + i}$, and if $a = a_i$ for some $i \geq \left\lfloor \frac{K+1}{2} \right\rfloor + 1$, then $b = a_{i - \left\lfloor \frac{K+1}{2} \right\rfloor}$.

2) If we denote by $M(a)$ the counter of a, then it holds that:

- *If $a < M(a)$ then $|\{b \in A: b \leq a \text{ or } M(a) \leq b\}| \geq (K+1)/2$*
- *If $M(a) < a$ then $|\{b \in A: b \leq M(a) \text{ or } a \leq b\}| \geq (K+1)/2$.*

Proof: The proof is given in Appendix E.

The next example illustrates the two statements of the lemma.

Example 4 *Let $A = \{a, b, c\}$ and assume that the preferences are single-caved with respect to the order $\leq$ such that: $a < b < c$, then $PC(\leq) = \{(abc), (acb), (cab), (cba)\}$. We can check that the counter of a is c and the counter of c is a, and that $[rank_{\succ}(a) < 2 \Leftrightarrow a \succ c]$ and $[rank_{\succ}(c) < 2 \Leftrightarrow c \succ a]$. We can check also that $|\{z \in A: z \leq a \text{ or } c \leq z\}| = |\{a, c\}| \geq (K+1)/2 = 2$*

The next claim shows that under single-caved profiles, if $\mathbb{C}$ is a *Condorcet committee à la Gehrlein*, then $\mathbb{C}$ is *"left continuous"* (i.e., for any member of $\mathbb{C}$ which is *"left to the middle"*, every other candidate which is *left to her* must be also in $\mathbb{C}$) and also it is *"right continuous"* (i.e., for any candidate in $\mathbb{C}$ which is *"right to the middle"*, every other candidate which is *right to her* must be also in $\mathbb{C}$).

Claim 4 *Let $\leq$ be a linear order on A and assume without loss of generality that $a_1 <\ldots< a_K$. Let π be a preference profile of single-caved preferences with respect to $\leq$, and let $\mathbb{C} = \{c_1, \ldots c_k\} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π and $|\mathbb{C}| < (K+1)/2$. Then:*

a) If $i < (K+1)/2$ and $a_i \in \mathbb{C}$ then $\forall j < i$ it holds that $a_j \in \mathbb{C}$ ("left continuous")

b) If $i > (K+1)/2$ and $a_i \in \mathbb{C}$ then $\forall j > i$ it holds that $a_j \in \mathbb{C}$ ("right continuous")

c) If $i = (K+1)/2$ then $a_i \notin \mathbb{C}$.

Proof: The proof is given in Appendix F.

The next Corollary shows that under single-caved profile, if $\mathbb{C}$ is a *Condorcet committee à la Gehrlein*, then for any candidate a in $\mathbb{C}$, her *counter* (denoted by $M(a)$) cannot be in $\mathbb{C}$.

Corollary 2 *Let $\leq$ be a linear order on A and assume without loss of generality that $a_1 < \ldots < a_K$. Let π be a preference profile of single-caved preferences with respect to $\leq$, and let $\mathbb{C} = \{c_1, \ldots, c_k\} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π and $|\mathbb{C}| < (K+1)/2$. Let a be any candidate such that $a \neq a_{(K+1)/2}$ and $M(a)$ is the counter of a (according to lemma 2). Then, it cannot be the case that $a \in \mathbb{C}$* and *$M(a) \in \mathbb{C}$.*

Proof: Assume by contradiction that $a \in \mathbb{C}$ and $(a) \in \mathbb{C}$. By Claim 4, $\mathbb{C}$ is *right continuous* and *left continuous*, consequently we obtain that: **If $\mathbf{a} < M(\mathbf{a})$** then $\{b \in A: b \leq a \text{ } or \text{ } M(a) \leq b\} \subseteq \mathbb{C}$ and $|\{b \in A: b \leq a \text{ } or \text{ } M(a) \leq b\}| \leq |\mathbb{C}|$, but by statement 2 of Lemma 2, $|\{b \in A: b \leq a \text{ } or \text{ } M(a) \leq b\}| \geq (K+1)/2$, which contradict the assumption that $|\mathbb{C}| < (K+1)/2$. **If $\boldsymbol{M(a) < a}$** then $\{b \in A: b \leq M(a) \text{ } or \text{ } a \leq b\} \subseteq \mathbb{C}$ and $|\{b \in A: b \leq M(a) \text{ } or \text{ } a \leq b\}| \leq |\mathbb{C}|$ but by statement 2 of Lemma 2, $|\{b \in A: b \leq M(a) \text{ } or \text{ } a \leq b\}| \geq (K+1)/2$, which also contradict the assumption that $|\mathbb{C}| < (K+1)/2$. ∎

The next Proposition shows that if the committee size does not exceed half the number of the candidates, and the preference profile is single-caved with respect to some linear order on A, then a *Condorcet committee à la Gehrlein* must be socially acceptable. However, when the committee size exceeds half the number of the candidates, then it may be socially acceptable or partly socially unacceptable, but it cannot be completely unacceptable.

Proposition 4 *Let $\leq$ be a linear order on A and assume without loss of generality that $a_1 < \ldots < a_K$. Let π be a single-caved preference profile with respect to $\leq$, and let $\mathbb{C} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π.*

1) If $|\mathbb{C}| < (K+1)/2$, then $\mathbb{C}$ is socially acceptable with respect to π.

2) If $|\mathbb{C}| \geq (K+1)/2$, then $\mathbb{C}$ can be socially acceptable with respect to π or partly unacceptable but it cannot be completely unacceptable.

Proof: **Assume $|\mathbb{C}| < (\boldsymbol{K}+\mathbf{1})/\mathbf{2}$.** If $a \in \mathbb{C}$ then by Claim 4 it must be that $a \neq a_{(K+1)/2}$. So, assume that there is $a \in \mathbb{C}$ such that $a \neq a_{(K+1)/2}$, then by Lemma 2 there is $b \in A$ (*it is $M(a)$ the counter* of a) such that for all $\succ \in PC(\leq)$, $rank_{\succ}(a) < (K+1)/2 \Leftrightarrow a \succ b$. Consequently, $\mu_\pi(\{\succ : rank_{\succ}(a) < (K+1)/2\}) = \mu_\pi(C(a \succ b))$. Since $a \in \mathbb{C}$, then by Corollary 2, $M(a) \notin \mathbb{C}$ (i.e., $b \notin \mathbb{C}$), and since $\mathbb{C}$ is a Condorcet committee, we must have that: $\mu_\pi\big(C(a \succ b)\big) > \mu_\pi\big(C(b \succ a)\big)$.

Hence, $\mu_\pi(C(a \succ b)) \geq n/2$. Therefore, $\mu_\pi(\{\succ: rank_\succ(a) < (K+1)/2\}) \geq n/2$. consequently, $\mu_\pi(\{\succ: rank_\succ(a) <) < (K+1)/2\}) \geq \mu_\pi(\{\succ: rank_\succ(a) >) < (K+1)/2\})$, namely a is socially acceptable. Consequently, $\mathbb{C}$ is socially acceptable with respect to π.

Now, assume $|\mathbb{C}| \geq (K+1)/2$. First, we show that at least one Condorcet committee member must be socially acceptable. This results from the proof of statement (1). Since π is single-caved, we obtain that a (weak) Condorcet winner, which exist, must be in $\mathbb{C}$, because otherwise $\mathbb{C}$ will be empty. Let a be a (weak) Condorcet winner. Since $a \in \mathbb{C}$ then by a previous argument in the proof, $a \neq a_{(K+1)/2}$, but then by Lemma 2, there is $b \in A$ such that $\forall \succ \in P(\leq)$, $rank_\succ(a) < (K+1)/2 \Leftrightarrow a \succ b$. Consequently, $\mu_\pi(\{\succ: rank_\succ(a) < (K+1)/2\}) \geq \mu_\pi(\{\succ: rank_\succ(a) > (K+1)/2\})$. That is, a is socially acceptable. Therefore, at least one member of $\mathbb{C}$ must be socially acceptable, that is $\mathbb{C}$ cannot be completely unacceptable with respect to π. The next example completes the proof and shows that a Condorcet committee can be socially acceptable or partly unacceptable with respect to single-caved profile.

Example 5 *Let $A = \{a, b, c\}$ and consider the following two single-caved profiles with respect to the order $a < b < c$: $\pi_1 = \{(abc), (acb), (acb)\}$, $\pi_2 = \{(abc), (cab), (cab)\}$. It can be checked that under π_1, $\{a, c\}$ is a Condorcet committee but it is partly unacceptable, and that under π_2, $\{a, c\}$ is a Condorcet committee and it is socially acceptable. Now, let $A = \{a, b, c, d\}$ and consider the following two single-caved profiles with respect to the order $a < b < c < d$: $\pi_1 = \{(abcd), (cabd),\}$, $\pi_2 = \{(abcd), (abcd), (abcd)\}$. It can be checked that under π_1, $\mathbb{C} = \{a, b, c\}$ is a Condorcet committee of size $|\mathbb{C}| \geq (K+1)/2$ and it is socially acceptable and that under π_2, $\mathbb{C} = \{a, b, c\}$ is a Condorcet committee but it is partly unacceptable (a, b are acceptable, c is unacceptable).Similar examples can be given for any K and $|\mathbb{C}| \geq (K+1)/2$.* ■

3.4 Condorcet committees under Single-crossing preferences

The concept of single-crossing preferences was introduced by Mirrlees (1971) and Roberts (1977). For other explanations, see for instance Saporiti and Fernando (2006). Informally speaking, a set of preferences on the candidates satisfy the single-crossing property if both these preferences and the candidates can be ordered from left to right so that if a rightist preference prefers a left candidate to a right candidate, then so do all preferences that are to the left of it. Formally,

Definition 8 Let A be the set of candidates and let $\leq$ be a linear order on A. Let $C \subseteq \mathcal{P}$ be a nonempty subset of preferences and let $\sqsubseteq$ be a linear order on C. We say that the preference relations in C *satisfy the single crossing property* with respect to $(\leq, \sqsubseteq)$ if for all pairs of candidates $a, b \in A$ and for all pairs of preferences $\succ, \succ' \in C$, we have

$$\{a < b \; and \; \succ \sqsubset \succ'\} \Rightarrow (b \succ a \Rightarrow b \succ' a)$$

If $a < b$ we say that a is *to the left* of b or that b is *to the right of* a, and if $\succ_1 \sqsubset \succ_2$ we say that $\succ_1$ is *to the left of* $\succ_2$ or that $\succ_2$ is *to the right of* $\succ_1$. We say that the profile π *satisfies the single crossing property* if there is a linear order $\leq$ on A and a linear order $\sqsubseteq$ on the set $\pi(N)$ of preferences in the profile, such that the preferences in $\pi(N)$ satisfy the single crossing property with respect to $(\leq, \sqsubseteq)$.

The next example illustrates the definition.

Example 6 *Let* $A = \{a, b, c, d, e\}$ *with the linear order given by* $a < b < c < d < e$. *Consider the subset* $C \subseteq \mathcal{P}$ *that contains the following* 6 *preference relations:* $\succ_1 = (abcde)$, $\succ_2 = (acbde)$,$\succ_3 = (acdbe)$, $\succ_4 = (adcbe)$, $\succ_5 = (adcbe)$, $\succ_6 = (adecb)$, *with the linear order on preference relations given by* $\succ_1 \sqsubset \succ_2 \sqsubset \succ_3 \sqsubset \succ_4 \sqsubset \succ_5 \sqsubset \succ_6$. *The preferences in* C *satisfy the single-crossing property with respect to* $(\leq, \sqsubseteq)$.

The next definition of *median preference* is useful for the following proposition.

Definition 9 Let $\leq$ be a linear order on A, and let $\sqsubseteq$ be a linear order on $\mathcal{P}$. Let π be a profile of preferences that satisfies the single-crossing property with respect to $(\leq, \sqsubseteq)$. We say that $\succ_m \in \pi(N)$ is a *median preference relation* of π if and only if

$$\mu_\pi(\{\succ \in \pi(N) : \succ \sqsubseteq \succ_m\}) \geq n/2 \text{ and } \mu_\pi(\{\succ \in \pi(N) : \succ_m \sqsubseteq \succ\}) \geq n/2.$$

Since the profile π satisfies the single-crossing property with respect to $(\leq, \sqsubseteq)$, the set of preferences $\pi(N)$ is ordered by $\sqsubseteq$ and a median relation $\succ_m \in \pi(N)$, always exists.

The next proposition states that if the preference profile satisfies the single-crossing property, then a Condorcet committee can be socially acceptable or partly unacceptable, but it cannot be completely unacceptable.

Proposition 5 *Let $\leq$ be a linear order on A and let $\sqsubseteq$ be a linear order on $\mathcal{P}$. Let π be satisfies the single-crossing property with respect to $(\leq, \sqsubseteq)$, and let $\mathbb{C} \in C(A)$ be a Condorcet committee à la Gehrlein with respect to π, then $\mathbb{C}$ can be socially acceptable with respect to π or partly unacceptable but it cannot be completely unacceptable.*

Proof: first we show that at least one Condorcet committee member must be socially acceptable. Since π satisfies the single-crossing property, there exists a median preference relation $\succ_m \in \pi(N)$. If we denote by a *the top candidate* of $\succ_m$ then a must be a (weak) Condorcet winner, i.e., $\mu_\pi\big(C(a \succ b)\big) \geq \mu_\pi\big(C(b \succ a)\big),\ \forall b \in A\backslash\{a\}\}$. It is true because:

- If $b < a$ then (since $a \succ_m b$) we obtain by single-crossing property that $a \succ_i b, \forall i$ such that $\succ_m \sqsubseteq \succ_i$, but these voters are at least $n/2$, thus $\mu_\pi\big(C(a \succ b)\big) \geq \mu_\pi\big(C(b \succ a)\big)$.
- If $a < b$ then (since $a \succ_m b$) we obtain (by single-crossing property) that $a \succ_i b, \forall i$ such that $\succ_i \sqsubseteq \succ_m$, but these voters are at least $n/2$, thus $\mu_\pi\big(C(a \succ b)\big) \geq \mu_\pi\big(C(b \succ a)\big)$.

Therefore, in either case a is preferred to b by at least $n/2$ voters. Now, by the definition of Condorcet committee à la Gehrlein, and since a is a (weak) Condorcet winner, it must be that $a \in \mathbb{C}$, because if $a \notin \mathbb{C}$, then any $b \in \mathbb{C}$ will not satisfy the requirement: $\mu_\pi\big(C(b \succ b')\big) > \mu_\pi\big(C(b' \succ b)\big)$, $\forall b' \in A\backslash\{b\}\}$, just if we take $b' = a$. By Lemma 1 in Mahajne and Volij (2018b), a must be socially acceptable. The next example completes the proof and shows that a Condorcet committee can be socially acceptable and can be partly unacceptable with respect to single-crossing profile.

Example 7 *Let $A = \{a, b, c\}$ and consider the following two single-crossing profiles with respect to the order $a < b < c$: $\pi_1 = \{(abc), (acb), (acb)\}$,$\pi_2 = \{(abc), (cab), (cab)\}$. It can be checked that under π_1, $\{a, c\}$ is a Condorcet committee that is partly unacceptable (a, b are acceptable, c is unacceptable), and that under π_2,$\{a, c\}$ is a Condorcet committee that is socially acceptable.*

Now, let $A = \{a, b, c, d\}$ and consider the following two single-crossing profiles with respect to the order $a < b < c < d$: $\pi_1 = \{(abcd), (abcd), (cabd), (cabd)\}$, $\pi_2 = \{(abcd), (abcd), (abcd)\}$. It can be checked that under π_1,$\mathbb{C} = \{a, b, c\}$ is a Condorcet committee that is socially acceptable and that under π_2, $\mathbb{C} = \{a, b, c\}$ is a Condorcet committee but it is partly unacceptable (a, b are acceptable, c is unacceptable). ∎

3.5 Condorcet committees under group-separable preferences

The concept of *group-separable* preferences was introduced by Inada (1964). A set of preference relations are *group-separable* with respect to single-crossing profile π if any subset (of size 3 at least) of the set of all candidates A can be partitioned into two disjoint non-empty subsets such that for each voter either she prefers every candidate from the first subset to every candidate from the second subset or she prefers every candidate from the second subset to every candidate from the first subset. Formally,

Definition 10 Let π be a preference profile. We say that π *satisfies the group-separable property* if any subset $B \subseteq A$ with $|\mathrm{B}| \geq 3$, can be partitioned into two disjoint non-empty subsets B_1 and B_2 such that for each voter $i \in N$, it holds that either $[\forall a \in \mathrm{B}_1 \text{ and } \forall b \in \mathrm{B}_2\colon\ a \succ_i b\ ]$ or $[\forall a \in \mathrm{B}_1 \text{ and } \forall b \in \mathrm{B}_2\colon\ b \succ_i a\ ]$. In this case we say that $\{\mathrm{B}_1, \mathrm{B}_2\}$ is a *group-separable partition* of B, and we say that the preference profile π is *group-separable*.

The next example illustrates the definition.

Example 8 *Let* $A = \{a, b, c, d\}$ and *consider the following profile of 5 voters:* $\{(abcd)$, $(abdc)$, $(acdb)$, $(cdba)$, $(bcda)\}$. *We can check that every subset of A (of size 3 at least) can be group-separable partitioned as follows:* $A = [\{a\}] \cup [\{b\} \cup \{c, d\}]$ (*for instance, candidate* a *is placed first or last by all the voters). Also, the committee* $\{a, b\}$ *is a Condorcet committee and is socially acceptable (each one of* $\{a, b\}$ *is above the line by 3 and below the line by 2).*

The next proposition, states that when there are more than three candidates, and the preference profile satisfies the *group-separable property*, then a Condorcet committee *à la* Gehrlein may be socially acceptable or partly unacceptable, but it cannot be completely unacceptable. However, when there are three candidates and the committee size is 1, then a Condorcet committee must be socially acceptable, but when the size is at least 2, then it can be socially acceptable or partly unacceptable but it cannot be completely unacceptable.

Proposition 6 *Let* π *be a group-separable preference profile, and let* $\mathbb{C} \in C(A)$ *be a Condorcet committee à la Gehrlein with respect to* π.

1) If $K = 3$, *then if* $k = 1$, $\mathbb{C}$ *must be socially acceptable with respect to* π *and if* $k = 2$, $\mathbb{C}$ *can be socially acceptable or partly unacceptable but it cannot be completely unacceptable.*

2) If $K \geq 4$, *then all the situations are possible.*

Proof: **Statement (1)**. If $k = 1$, then $\mathbb{C} = \{a\}$, where a is a (strong) Condorcet winner, which exists (otherwise $\mathbb{C}$ will be empty). By the first statement of Proposition 1, the committee $\mathbb{C} = \{a\}$ is socially acceptable. If $k = 2$, we have (w.l.g) that $\mathbb{C} = \{a, b\}$. In this case, one of them (a w.l.g) must be (weak) Condorcet winner, i.e., $\mu_\pi(C(a \succ a')) \geq \mu_\pi(C(a' \succ a))\ \ \forall a' \in A\backslash\{a\}$ (otherwise, $\mathbb{C}$ will be empty). But, by the proof of the first statement of Proposition 1, a must be socially acceptable. Therefore, at least one member of $\mathbb{C}$ is socially acceptable. The next example completes the proof of the statement and shows that for $k = 2$, $\mathbb{C}$ *can be acceptable or partly unacceptable:*

Example 9 *Let* $A = \{a, b, c\}$ *and consider the following group-separable profiles:* $\pi_1 = \{(abc), (cab), (cab)\}$ *and* $\pi_2 = \{(abc), (acb), (acb)$. *These profiles are group-separable: for* π_1, $\{\{a, b\}, \{c\}\}$ is a group-separable partition of A, and for π_2, $\{\{a\}, \{b, c\}\}$ is a group-separable partition of A. *It can be seen that under* π_1, *{a,c} is a Condorcet committee and it is socially acceptable, and under* π_2, *{a,c} is a Condorcet committee but it is partly unacceptable.*

Statement (2) The next example completes the proof and shows that when $K \geq 4$, all the situations are possible.

Example 10 *Let A={a, b, c, d} and let k=1. Consider the following group-separable profile of 15 voters: 4 voters with preference* $(abcd)$, *2 voters with* $(bacd)$, *2 voters with* $(cbad)$, *3 voters with* $(dbac)$, *4 voters with* $(dcab)$. *It can be checked that under this profile, {a} is a Condorcet committee and it is completely unacceptable.*

Now let k=2. Consider the following group-separable profiles: $\pi_1 = \{(abcd), (abcd), (abcd)\}$, $\pi_2 = \{(abcd), (acbd), (bcad), (bcad), (dabc), (dacb), (dcba)\}$. *Indeed these profiles are group-separable since* $\{\{\{a\}, \{b, c\}\}, \{d\}\}$ is a group-separable partition of A. *It can be checked that under* π_1, *{a,b} is a Condorcet committee and it is socially acceptable, and that under* π_2, *{a,b} is a Condorcet committee and it is partly unacceptable.*

Consider also the following profile of 17 voters: 4 voters with preference $(abcd)$, *2 voters with* $(bacd)$, *2 voters with* $(cbad)$, *1 voter with* $(dabc)$, *3 voters with* $(dbac)$, *4 voters with* $(dcab)$, *1 voter with* $(dcba)$. *This profile is group-separable since* $\{\{\{a, b\}, \{c\}\}, \{d\}\}$ is a group-separable partition of A. *It can be checked that under this profile, {a,d} is a Condorcet committee but it is completely unacceptable. Similar examples can be given for any* $K > 4$. ■

4. Probability evaluation of the social acceptability of Condorcet committees

In this section, we evaluate the probability of having a socially acceptable or a completely/partly unacceptable Condorcet committee. We consider the elementary assumption called the Impartial Anonymous Culture (IAC). With K candidates, there are $K!$ possible strict orderings and a voting situation is defined by the vector $\tilde{n} = (n^1, n^2, \dots, n^j, \dots n^{K!})$ such that $\sum_{j=1}^{j=K!} n^j = n$. The integer $n^{\mathrm{j}} = |\{i \in N: \succ_i = \succ^{\mathrm{j}}\}|$ being the number of voters endowed with the j^{th} corresponding linear order and recall that n is the total number of voters. The IAC condition stipulates that all voting situations $\tilde{n}$ are equally likely to be observed. In other words, all combinations of n^j that sum to a specified n are equiprobable. This assumption, introduced by Gehrlein and Fishburn (1976) is one of the most used hypothesis in the literature of social choice theory when computing the theoretical likelihood of electoral events. For more details on the IAC condition and others, we refer the reader to Gehrlein and Lepelley (2011, 2017).

Let Pr_1, Pr_2, and Pr_3 the probability for a Condorcet committee, when it exists, to be partly unacceptable, completely unacceptable, and socially acceptable, respectively. If necessary, for Pr_1, we will use the notation $Pr_1(u, \bar{u})$ with u is the number of unacceptable members and $\bar{u}$ is the number of acceptable members in the considered Condorcet committee. Naturally, $u + \bar{u} = k$.

4.1 Three candidates

For three-candidate elections, Proposition 1 shows that every Condorcet committee of size one (Condorcet winner), when it exists, is socially acceptable and a Condorcet committee of size two, when it exists, can never be socially completely unacceptable. The results for the other situations with three-candidate elections are described in Propositions 7 and 8. The proof of each proposition can be formulated as counting the exact number of integer solutions in finite systems of linear constraints with rational coefficients. As pointed out by Lepelley et al. (2008) and Wilson and Pritchard (2007), Ehrhart polynomials (Ehrhart, 1962, 1967) are the appropriate mathematical concepts to study such problems.

Proposition 7 *The probability Pr_1 for a Condorcet committee of size two, when it exists, to be socially partly unacceptable under IAC condition in three-candidate elections is given by:*

- $\frac{n^3+3n^2-16}{2(n+4)(n+2)^2}$ $if\ n \equiv 0\ mod\ 4$
- $\frac{(n+4)(n-1)}{2(n+5)(n+1)}$ $if\ n \equiv 1\ mod\ 4$
- $\frac{(n+3)(n^2-4)}{2n(n+4)^2}$ $if\ n \equiv 2\ mod\ 4$
- $\frac{n^3+6n^2+9n-12}{2(n+3)^3}$ $if\ n \equiv 3\ mod\ 4$

Proof: In order to obtain a representation for the probability Pr_1that a Condorcet committee of size two, when it exists, is socially partly unacceptable under IAC condition, we need to calculate two different values: the number of voting situations for which a Condorcet committee of size two exists and the number of voting situations for which a Condorcet committee of size two is socially partly unacceptable.

We begin first by calculating the number of voting situations for which a Condorcet committee of size two exists. Let $\succ^1=$ (abc), $\succ^2=$ (acb), $\succ^3=$ (bac), $\succ^4=$ (cab), $\succ^5=$ (bca), $\succ^6=$ (cba). Thus, a voting situation is defined by $\tilde{n} = (n^1, \dots, n^6)$ such that $\sum_{j=1}^{j=6} n^j = n$. Suppose (w.l.g) that $\mathbb{C} = \{a, b\}$. According to Definition 4, this is equivalent to: $n^1 + n^2 + n^3 - n^4 - n^5 - n^6 > 0$ (i) and $n^1 - n^2 + n^3 - n^4 + n^5 - n^6 > 0$ (ii). Using the Parameterized Barvinok's[15] (see for instance Verdoolaege et al., 2004), the number of integer points inside the polytope defined by (i), (ii), $\sum_{j=1}^{j=6} n^j = n$, and $n^j \geq 0$ is given by the following 2-periodic Ehrhart polynomial:

$$E(n) = \frac{1}{384}n^5 + \left[\frac{1}{32}, \frac{5}{128}\right]_n n^4 + \left[\frac{13}{96}, \frac{43}{192}\right]_n n^3 + \left[\frac{1}{4}, \frac{39}{64}\right]_n n^2 + \left[\frac{1}{6}, \frac{99}{128}\right]_n n + \left[0, \frac{45}{128}\right]_n$$

The bracketed list $\left[\frac{1}{32}, \frac{5}{128}\right]_n$, for instance, is a 2-periodic number meaning that it depends on the remainder after division of n by 2: it is equivalent to $\frac{1}{32}$ if $n \equiv 0\ mod\ 2$ and $\frac{5}{128}$ if $n \equiv 1\ mod\ 2$.

The second step is then to introduce the conditions under which the Condorcet committee $\mathbb{C} = \{a, b\}$ is socially partly unacceptable. Two cases are possible: either a is socially acceptable whereas b is socially unacceptable or the opposite case. Using the symmetry of IAC with respect to the three candidates, these two cases are similar. Consider (w.l.g) the first case which is equivalent to two additional inequalities applying Definitions 1 and 2: $n^1 + n^2 \geq n^3 + n^4$(iii) and $n^1 + n^2 < n^3 + n^5$(iv). Using again the Parameterized Barvinok's algorithm, the number of integer points inside the

[15] The free software to calculate the integer points inside polytopes with the Parameterized Barvinok's algorithm can be found in http://freecode.com/projects/barvinok.

polytope defined by (i), (ii), (iii), (iv), $\sum_{j=1}^{j=6} n^j = n$, and $n^j \geq 0$ is given by the 4-periodic Ehrhart polynomial:

$$I(n) = \frac{1}{1536}n^5 + \left[\frac{7}{1536},\frac{1}{128}\right]_n n^4 + \left[\frac{1}{128},\frac{25}{768}\right]_n n^3 + \left[\frac{-1}{96},\frac{3}{64}\right]_n n^2 + \left[\frac{-1}{24},\frac{-9}{512}\right]_n + \left[0,\frac{-9}{128},\frac{-1}{32},\frac{-5}{128}\right]_n$$

As a consequence, the probability Pr_1 is defined by $2 \times I(n)$ divided by $E(n)$. This proves Proposition 7. ■

Proposition 8 *The probability Pr_3 for a Condorcet committee of size two, when it exists, to be socially acceptable under IAC condition in three-candidate elections is given by:*

- $\frac{n^3+13n^2+40n+48}{2(n+4)(n+2)^2}$ $if\ n \equiv 0\ mod\ 4$
- $\frac{(n+7)(n+2)}{2(n+5)(n+1)}$ $if\ n \equiv 1\ mod\ 4$
- $\frac{n^3+13n^2+36n+12)}{2n(n+4)^2}$ $if\ n \equiv 2\ mod\ 4$
- $\frac{n^3+12n^2+45n+66}{2(n+3)^3}$ $if\ n \equiv 3\ mod\ 4$

Proof: To find a representation for Pr_3 we use the same methodology that was developed in the proof of Proposition 7. Suppose again (w.l.g) that $\mathbb{C} = \{a, b\}$. This defines the conditions (i) and (ii) previously given in the proof of Proposition 7 and it has been found that the number of integer points inside this polytope is given by $E(n)$. According to Definitions 1 and 2, in order for this Condorcet committee to be socially acceptable, the two following additional inequalities have to be fulfilled: $n^1 + n^2 \geq n^3 + n^4$ (v) and $n^1 + n^2 \geq n^3 + n^4$ (vi). Using again the Parameterized Barvinok's algorithm, the number of integer points inside the polytope described by (i), (ii), (v), (vi), $\sum_{j=1}^{j=6} n^j = n$, and $n^j \geq 0$ is given by the 4-periodic Ehrhart polynomial:

$$G(n) = \frac{1}{768}n^5 + \left[\frac{17}{768},\frac{3}{128}\right]_n n^4 + \left[\frac{23}{192},\frac{61}{384}\right]_n n^3 + \left[\frac{13}{48},\frac{33}{64}\right]_n n^2 + \left[\frac{1}{4},\frac{207}{256}\right]_n + \left[0,\frac{63}{128},\frac{1}{16},\frac{55}{128}\right]_n$$

Finally, the probability Pr_3 is defined by $G(n)$ divided by $E(n)$. This proves Proposition 8. ■

The results of our computations for three-candidate elections are provided in Table 1. We observe that the probability Pr_1 is not negligible even for small electorates. We also observe that this probability should decrease as the number of voters increases. The probability Pr_1 represents half of the voting situations for large electorates.

Table 1: K = 3 and k = 2

n	Pr_1	Pr_2	Pr_3
3	0.22222	0	0.77778
4	0.16667	0	0.83333
5	0.30000	0	0.70000
6	0.24000	0	0.76000
7	0.34400	0	0.65600
8	0.28667	0	0.71333
9	0.37143	0	0.62857
10	0.31837	0	0.68163
50	0.45366	0	0.54634
51	0.47218	0	0.52782
100	0.47596	0	0.52404
101	0.48557	0	0.51443
1000	0.49751	0	0.50249
1001	0.49851	0	0.50149
∞	0.50000	0	0.50000

4.2 More than three candidates

As noticed by Lepelley et al. (2008), the parameterized Barvinok's algorithm that can be used in the three-candidate elections does not allow to deal with four candidates and more. However, recent developments within the polytope theory allow us to obtain exact results for the case of four candidates with the algorithm Normaliz (Bruns et al., 2017a) which is, to the best of our knowledge, the only program able to compute the number of voting situations in polytopes corresponding to elections with up to four candidates. The reader is referred to Bruns et al. (2017b) who describe several results obtained in four-candidate elections with Normaliz. Notice that this software needs relatively high memory when the number of voters increases. Consequently, even for the case of four candidates, we can only obtain exact results with small number of voters $3 \leq n \leq 10$. We also obtain exact results for four candidates with infinite set of voters using another method. Indeed, it is well known from the literature that the calculations of the limiting probability under IAC condition are simply reduced to computation of volumes of convex polytopes. For this purpose, our volumes are found with the use of the algorithm Convex which is a Maple package for convex geometry Franz (2017). The package works with the same procedure that was at first used in Cervone et al. (2005) and recently used in other studies such as Diss and Doghmi (2016), Diss and Gehrlein (2012, 2015), Gehrlein etal. (2015), and Moyouwou and Tchantcho (2017).For all remaining calculations, that is $K = 4$ with a finite number of voters $n > 10$ as well as $K = 5$ and $K = 6$, computer simulations are used. We describe in the following the Monte-Carlo simulation methodology that we apply in order to estimate our probabilities. As an illustration let us consider the probability Pr_3 of social acceptability of a Condorcet committee of size k.

- Step 1: At the beginning of the simulation, we randomly generate a voting situation of length $K!$.
- Step 2: We check whether the considered voting situation of step 1 generates a Condorcet committee of size k or not.
- Step 3: If a Condorcet committee of size k exists in step 2, we check whether the conditions of social acceptability are fulfilled or not.
- Step 4: These three steps are iterated 1,000,000 times.
- Step 5: The probability of social acceptability is then calculated as the quotient of the number of voting situations for which a Condorcet committee of size k, when it exists, is acceptable divided by the number of voting situations for which a Condorcet committee of size k exists.

The results of our computations are provided in Appendix G. Notice that the probability values 0, 1, 0.00000, and 1.00000 have different meanings. The first two correspond to exact values while the last two values are obtained using our simulation method. First, our results show that the probability for a Condorcet committee to be completely unacceptable (Pr_2) is quite negligible in all cases. Second, our results indicate that the probability Pr_1 is particularly high when the size of the committee is equal or exceeds half the number of the candidates. Clearly, social unacceptability may occur significantly. For instance, when there are six candidates, more than 80% of the voting situations generating a Condorcet committee of size four are expected to be partly unacceptable.

5. Concluding remarks

In this paper, we have examined the social acceptability of Condorcet committees in general and particularly under some types of restricted preference domains. As previously noticed, we have considered four types which are the most extensively studied in the literature of social choice theory. However, we believe that studying other restriction domains is an important research direction. We refer the reader to, for instance, the domain of value-restricted preference profiles (introduced by Sen, 1966), or the domain of level *r* consensus profiles (introduced by Mahajne et al., 2015).

A second line of research which is already started by the two authors is to examine and compare multi-winner voting rules according to social acceptability criterion. We mention some of the common rules that deserve a careful consideration: The k-Plurality rule returns *k* candidates with the highest Plurality scores, the Bloc rule returns *k* candidates with the highest *k*-approval scores, the *k*-Borda rule returns *k* candidates with the highest Borda scores, and the Chamberlin–Courant and Monroe rules which aim at proportional representation (Elkind et al., 2017; Faliszewski, et al., 2015).

Finally, it is important to stress that obtaining probability results with more than six candidates or other assumptions about the nature or distribution of individual preferences can also be an important line of research. However, this option is ignored since the conclusions of our paper clearly show that Condorcet committees are exposed to social unacceptability to a significant extent.

References

Aziz, H., Elkind, E., Faliszewski, P., Lackner, M., and Skowron, P. (2017) The Condorcet principle for multiwinner election: from short listing to proportionality. Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence (IJCAI-17), 84-90.

Baharad, E., and Nitzan, S. (2003). The Borda rule, Condorcet consistency and Condorcet stability. Economic Theory 22, 685-688.

Barberà, S., and Coelho, D. (2008) How to choose a non-controversial list with *k* names. Social Choice and Welfare 31: 79-96.

Black, D. (1948) On the Rationale of Group Decision Making. The Journal of Political Economy 56: 23–34.

Bruns, W., Ichim, B., Römer, T., Sieg, R., and Söger, C. (2017a) Normaliz: Algorithms for rational cones and affine monoids. Available at http://normaliz.uos.de.

Bruns, W., Ichim, B., and Söger, C. (2017b) Computations of volumes and Ehrhart series in four candidates elections. Working paper, Cornell University Library.

Cervone, D., Gehrlein, W.V., and Zwicker, W. (2005) Which scoring rule maximizes Condorcet efficiency under IAC? Theory and Decision 58: 145-185.

Coelho, D. (2004) Understanding, evaluating and selecting voting rules through games and axioms (Ph.d. thesis). UniversitatAutonoma de Barcelona, Departament d'Economia i d'Historia Economica. Available at http://hdl.handle.net/10803/4056.

Courtin S., Martin M., and Moyouwou, I. (2015a) The *q*-majority efficiency of positional rules. Theory and Decision 79(1): 31-49.

Courtin S., Martin M., and Tchantcho, B. (2015b) Positional rules and *q*-Condorcet consistency. Review of Economic Design 19(3): 229-245.

Diss, M., and Doghmi, A. (2016) Multi-winner scoring election methods: Condorcet consistency and paradoxes. Public Choice 169: 97-116

Diss, M., and Gehrlein, W.V. (2015) The true impact of voting rule selection on Condorcet efficiency. Economics Bulletin 35: 2418-2426.

Diss, M., and Gehrlein, W.V. (2012) Borda's paradox with weighted scoring rules. Social Choice and Welfare 38: 121-136.

Diss, M., Kamwa, E. and Tlidi, A. (2018) The Chamberlin-Courant Rule and the k-Scoring Rules: Agreement and Condorcet Committee Consistency. Mimeo.

Ehrhart, E. (1962) Sur les polyèdres rationnels homothétiques à *n* dimensions. Comptes Rendus de l'Academie des Sciences, Paris, 254: 616-618.

Ehrhart, E. (1967) Sur un problème de géométrie diophantienne linéaire. Ph.D. Thesis. Journal für die Reine und Angewandte Mathematik 226: 1-49.

Elkind, E., Lang, J., and Saffidine, A. (2015) Condorcet winning sets. Social Choice and Welfare 44(3): 493-517.

Elkind, E., Faliszewski, P., Skowron, P. and Slinko, A. (2017) Properties of multiwinner voting rules, Social Choice and Welfare 48(3): 599–632.

Faliszewski, P., Skowron, P, Slinko, A., and Talmon, N. Multiwinner voting: A new challenge for social choice theory. In U. Endriss, editor, Trends in Computational Social Choice, pages 27–47. AI Access, 2017.

Franz, M. (2017) Convex - a Maple package for convex geometry, version 1.2 available at http://www- home.math.uwo.ca/ mfranz/convex/.

Gehrlein, W. (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences 10: 199-209.

Gehrlein, W.V., and Fishburn, P.C. (1976) The probability of the paradox of voting: A computable solution. Journal of Economic Theory 13: 14-25.

Gehrlein, W.V., and Lepelley, D. (2011) Voting paradoxes and group coherence. Springer, Berlin/Heidelberg.

Gehrlein, W.V., and Lepelley, D. (2017) Elections, voting rules and paradoxical outcomes. Springer-Verlag.

Gehrlein, W.V., Lepelley, D., and Moyouwou, I. (2015) Voters' preference diversity, concepts of agreement and Condorcet's paradox. Quality & Quantity 49(6): 2345-2368.

Inada, K. (1964) A note on the simple majority decision rule. Econometrica, 32:525–531.

Inada, K. (1969) The simple majority rule. Econometrica 37:490–506.

Kamwa, E. (2017) On stable rules for selecting committees. Journal of Mathematical Economics 70: 36-44.

Lepelley, D., Louichi, A., and Smaoui, H. (2008) On Ehrhart polynomials and probability calculations in voting theory. Social Choice and Welfare 30: 363-383.

Mahajne, M. and Volij, O. (2018a) The socially acceptable scoring rule. Social Choice and Welfare 51(2): 223-233.

Mahajne, M. and Volij, O. (2018b) Condorcet winners and socially acceptability. Mimeo.

Mahajne, M. and Volij, O. (2017) Consensus and single-peakedness. Mimeo.

Mahajne, M., Nitzan, S., and Volij, O. (2015) Level *r* Consensus and Stable Social Choice. Social Choice and Welfare 45(4): 805-817.

Mirrlees, J. (1971), An Exploration in the Theory of Optimal Income Taxation. Review ofEconomic Studies 38: 175–208.

Moyouwou, I., and Tchantcho, H. (2017) A note on Approval Voting and electing the Condorcet loser. Mathematical Social Sciences 89: 70-82.

Ratliff, T. (2003) Some startling inconsistencies when electing committees. Social Choice and Welfare 21: 433-454.

Roberts, K. (1977), Voting Over Income Tax Schedules. Journal of Public Economics 8: 329–340.

Saporiti, A., and Fernando, T. (2006) Single-crossing, strategic voting and the median choice rule. Social Choice and Welfare 26: 363–383.

Sen, A. K. (1966) A possibility theorem on majoritydecisions. Econometrica 34: 491-499.

Verdoolaege, S., Woods, K.M., Bruynooghe, M., and Cools, R. (2005) Computation and manipulation of enumerators of integer projections of parametric polytopes. Technical Report CW 392, Katholieke Universiteit Leuven.

Wilson, M.C., and Pritchard, G. (2007) Probability calculations under the IAC hypothesis. Mathematical Social Sciences 54: 244-256.

Appendix

A. Proof of Claim 1

Let $\succ$ be a single-peaked with respect to $\leq$, then there is $p \in A$ such that

$$\forall x, y \in A: (x < y \leq p \text{ or } p \leq y < x) \Rightarrow y \succ x.$$

There are 3 cases for p: **Case 1**:$a < p < b$. There are *two situations:* $a < c \leq p < b$ and $a < p \leq c < b$. In the *first situation*, if $a \succ b$ then (by single-peakedness, and $a < c \leq p$) we have also $c \succ a$, consequently, $c \succ b$, and if $b \succ a$ then we have as before $c \succ a$. In the *second situation*, if $a \succ b$, then (by single-peakedness and $p \leq c < b$) we have also $c \succ b$, and if $b \succ a$ then we have as before $c \succ b$, and consequently $c \succ a$. **Case 2**: $p \leq a$. In this case we obtain that $p \leq a < c < b$, thus it must be (by single-peakedness) that $a \succ b$ and $c \succ b$. **Case 3**: $p \geq b$. In this case we obtain that $a < c < b \leq p$, thus it must be (by single-peakedness) that $b \succ a$ and $c \succ a$.

Therefore, in all cases we obtain: $(a \succ b) \Rightarrow (c \succ b)$ and $(b \succ a) \Rightarrow (c \succ a)$. Now, if $[(a \succ b) \Rightarrow (c \succ b)]$, then $C(a \succ b) \subseteq C(c \succ b)$. Hence, $\mu_\pi(C(a \succ b)) \leq \mu_\pi(C(c \succ b))$. Similarly we obtain also that $\mu_\pi(C(b \succ a)) \leq \mu_\pi(C(c \succ a))$. ■

B. Proof of Claim 2

For any $x, y \in A$, let $\Delta\mu_\pi(x \succ y) = \mu_\pi(C(x \succ y)) - \mu_\pi(C(y \succ x))$. Assume by contradiction that in A there is $c \notin \mathbb{C}$ such that $a < c < b$ for some $a, b \in \mathbb{C}$. By Claim 1, if $\succ$ is single-peaked with respect to $\leq$ then: $[(a \succ b) \Rightarrow (c \succ b)]$, and $[(b \succ a) \Rightarrow (c \succ a)]$.

Hence, $\mu_\pi(C(a \succ b)) \leq \mu_\pi(C(c \succ b))$, and therefore, $\Delta\mu_\pi(a \succ b) \leq \Delta\mu_\pi(c \succ b)$ (1)

Also, $\mu_\pi(C(b \succ a)) \leq \mu_\pi(C(c \succ a))$, and therefore, $\Delta\mu_\pi(b \succ a) \leq \Delta\mu_\pi(c \succ a)$ (2)

▪ If $\Delta\mu_\pi(a \succ b) \leq 0$ then (conversely) $\Delta\mu_\pi(b \succ a) \geq 0$, and thus by (2), $\Delta\mu_\pi(c \succ a) \geq o$, but this contradicts the assumption that $\mathbb{C}$ is a Condorcet committee and $a \in \mathbb{C}$ and $c \notin \mathbb{C}$, which requires that $\Delta\mu_\pi(c \succ a) < o$.

▪ If $\Delta\mu_\pi(a \succ b) \geq 0$, and thus by (1) $\Delta\mu_\pi(c \succ b) \geq o$, but this also contradicts the assumption that $\mathbb{C}$ is a Condorcet committee and $b \in \mathbb{C}$ and $c \notin \mathbb{C}$, which requires that $\Delta\mu_\pi(c \succ b) < o$. ■

C. Proof of Lemma 1

The proof of **statement** 1 results from Theorem 1 in Mahajne and Volij (2018b). However, we present a similar proof with some changes, for the use of statement 2. Let $a \neq a_{(K+1)/2}$.

Case 1: $a = a_i$ for some $i \leq \left\lceil\frac{K-1}{2}\right\rceil$. Let $b = a_{\left\lceil\frac{K-1}{2}\right\rceil+i}$.

▪ **If** a and b are both placed above the line by some preference relation $\succ$, then (by single-peakedness) $\forall c \in A$ such that $a \leq c \leq b$, c is also placed above the line by $\succ$. But then it would be at least $(\left\lceil\frac{K-1}{2}\right\rceil + i) - i + 1$ candidates above the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible. (i.e., if a is placed above the line then b must be placed below or on the line and consequently, $a \succ b$**)**

▪ **If** a and b are both placed below the line by some $\succ$, then (by single-peakedness) it must be that, $\forall c \in A$ such that $c \leq a$ and $\forall c \in A$ such that $b \leq c$, c is also below the line by $\succ$. But, then it would be at least $i + (K - \left\lceil\frac{K-1}{2}\right\rceil - i + 1)$ candidates below the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible.

▪ **If** a is on the line by some $\succ$ (in this case K is odd), then b must be above the line by $\succ$ (and then $b \succ a$), because otherwise there are 2 sub-cases:

1: $a < p(\succ)$ and then, $\forall c \in A$ such that $c < a$ and $\forall c \in A$ such that $b \leq c$, c is also below the line. But, then it would be at least $(i-1) + (K - \left\lceil\frac{K-1}{2}\right\rceil - i + 1) = K - \left\lceil\frac{K-1}{2}\right\rceil$ candidates below the line, which is more than $(K-1)/2$ but this is impossible.

2: $p(\succ) < a$ and then,$\forall c \in A$ such that $a < c$, c is also below the line. But, then at least $(K-i)$ candidates will be below the line, which (since $i \leq \left\lceil\frac{K-1}{2}\right\rceil$) is more than $\left\lceil\frac{K-1}{2}\right\rceil$ but this is impossible.

As a result, we conclude that, $\succ$ places a above the line if and only if $a \succ b$.

Case 2: $a = a_i$ for some $i \geq \left\lceil\frac{K+1}{2}\right\rceil + 1$. Let $b = a_{i-\left\lceil\frac{K-1}{2}\right\rceil}$.

▪ **If** a and b are both placed above the line by some $\succ$, then (by single-peakedness) $\forall c \in A$ such that $b \leq c \leq a$ it must be that c is also above the line by $\succ$. But then at least $(i - (i - \left\lceil\frac{K-1}{2}\right\rceil)) + 1$ candidates will be above the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible. (i.e., if a is placed above the line then b must be placed below or on the line and consequently, $a \succ b$**)**

▪ **If** a and b are both placed below the line by some $\succ$, then (by single-peakedness), $\forall c \in A$ such that $c \leq b$ and $\forall c \in A$ such that $a \leq c$, c is also below the line. But, then at it would be that least

$(K-i+1)+(i-\lceil\frac{K-1}{2}\rceil)=K-\lceil\frac{K-1}{2}\rceil+1$ candidates are below the line which is more than $\lceil\frac{K-1}{2}\rceil$, but this is impossible.

▪ **If** a is on the line by some $\succ$ (in this case K is odd), then b must be above the line by $\succ$ (and then $b \succ a$), because otherwise there are 2 sub-cases:

1: $a > p(\succ)$ and then, $\forall c \in A$ such that $c > a$ and $\forall c \in A$ such that $c \leq b$, c is also below the line. But, then it would be at least $(K-i)+(i-\lceil\frac{K-1}{2}\rceil)=K-\lceil\frac{K-1}{2}\rceil$ candidates are below the line, which is more than $\lceil\frac{K-1}{2}\rceil$ but this is impossible.

2: $a < p(\succ)$ and then,$\forall c \in A$ such that $c < a$, c is also below the line. But, then it would be at least $(i-1)$ candidates are below the line, which (since $i \geq \lceil\frac{K+1}{2}\rceil+1$) is more than $\lceil\frac{K-1}{2}\rceil$ but this is impossible.

As a result, we conclude that, $\succ$ places a above the line if and only if $a \succ b$.

We now prove **Statement 2**. By construction of the proof of Statement 1, we obtain that:

▪ If $a < M(a)$ then $a = a_i$ for some $i \leq \lceil\frac{K-1}{2}\rceil$ and its *counter* $(M(a))$ *is* $b = a_{\lceil\frac{K-1}{2}\rceil+i}$. Therefore, there are $\lceil\frac{K-1}{2}\rceil+i-i+1=\lceil\frac{K-1}{2}\rceil+1$ candidates in $\{b \in A: a \leq b \leq M(a)\}$ which is at least $(K+1)/2$.

▪ If $M(a) < a$ then $a = a_i$ for some $i \geq \lceil\frac{K+1}{2}\rceil+1$ and then its *counter* $(M(a))$ *is* $b = a_{i-\lceil\frac{K-1}{2}\rceil}$. Therefore, there are $i-(i-\lceil\frac{K-1}{2}\rceil)+1=\lceil\frac{K-1}{2}\rceil+1$ candidates in $\{b \in A: M(a) \leq b \leq a\}$ which is at least $(K+1)/2$.

In both cases we would have at least $(K+1)/2$ candidates in $\{b \in A: a \leq b \leq M(a)\}$ (or in $\{b \in A: M(a) \leq b \leq a\}$). ■

D. Proof of Claim 3

The proof that $(b \succ a) \Longrightarrow (c \succ b)$ is immediate by the definition of single-caved preference relation and it is left to the reader, thus we obtain: $\{\succ \in \pi(N): b \succ a\} \subseteq \{\succ \in \pi(N): c \succ b\}$. Consequently, $\mu_\pi(C(b \succ a) \leq \mu_\pi(C(c \succ b))$. Similarly, we obtain that $\mu_\pi(C(b \succ c) \leq \mu_\pi\big(C(a \succ b)\big)$. ■

E. Proof of Lemma 2

First, we prove **statement 1**. Let $a \neq a_{(K+1)/2}$.

Case 1: $a = a_i$ for some $i \leq \left\lceil\frac{K-1}{2}\right\rceil$. Let $b = a_{\left\lfloor\frac{K+1}{2}\right\rfloor+i}$.

▪ **If** a and b are both placed above the line by some preference relation $\succ$, then (by single-cavedness) $\forall c \in A$ such that $c \leq a$ and $\forall c \in A$ such that $b \leq c$, c is also above the line by $\succ$. But then it would be at least $(i + (K - \left\lfloor\frac{K+1}{2}\right\rfloor - i + 1) = \left\lceil\frac{K-1}{2}\right\rceil + 1$ candidates above the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible. (i.e., if a is placed above the line then b must be placed below or on the line and consequently, $a \succ b$**)**

▪ **If** a and b are both placed below the line by some $\succ$, then (by single-cavedness) it must be that, $\forall c \in A$ such that $a \leq c \leq b$ it must be that c is also below the line. But, then it would be at least $(\left\lfloor\frac{K+1}{2}\right\rfloor + i - i + 1)$ candidates below the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible.

▪ **If** a is on the line by some $\succ$ (in this case K is odd), then b must be above the line by $\succ$ (and then $b \succ a$), because otherwise $\forall c \in A$ such that $a < c \leq b$, c is also below the line. But, then it would be at least $(\left\lfloor\frac{K+1}{2}\right\rfloor + i - i) = \left\lfloor\frac{K+1}{2}\right\rfloor$ candidates below the line, which is more than $\left\lceil\frac{K-1}{2}\right\rceil$ but this is impossible.

As a result, we conclude that $\succ$ places a above the line if and only if $a \succ b$.

Case 2: $a = a_i$ for some $i \geq \left\lfloor\frac{K+1}{2}\right\rfloor + 1$. Let $b = a_{i-\left\lfloor\frac{K+1}{2}\right\rfloor}$.

▪ **If** a and b are both placed above the line by some $\succ$, then (by single-cavedness) $\forall c \in A$ such that $a \leq c$ and $\forall c \in A$ such that $c \leq b$, c is also above the line by $\succ$. But then it would be at least $(K - i + 1) + (i - \left\lfloor\frac{K+1}{2}\right\rfloor) = K - \left\lfloor\frac{K+1}{2}\right\rfloor + 1$ candidates above the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible. (i.e., if a is placed above the line then b must be placed below or on the line and consequently, $a \succ b$**)**

▪ **If** a and b are both placed below the line by some $\succ$, then (by single-cavedness) $\forall c \in A$ such that $b \leq c \leq a$, c is also below the line by $\succ$. But, then it would be at least $(i - \left(i - \left\lfloor\frac{K+1}{2}\right\rfloor\right) + 1) = \left\lfloor\frac{K+1}{2}\right\rfloor + 1$ candidates below the line which is more than $\left\lceil\frac{K-1}{2}\right\rceil$, but this is impossible.

▪ **If** a is on the line by some $\succ$ (in this case K is odd), then b must be above the line, because otherwise $\forall c \in A$ such that $b \leq c < a$, c is must be also below the line by $\succ$. But, then it would be at least $(i - (i - \left\lfloor\frac{K+1}{2}\right\rfloor) = \left\lfloor\frac{K+1}{2}\right\rfloor$ candidates are below the line, which is more than $\left\lceil\frac{K-1}{2}\right\rceil$ but this is impossible.

We now prove **Statement 2**. By construction of the proof of Statement 1, we obtain that:

- If $a < M(a)$ then $a = a_i$ for some $i \leq \left\lceil\frac{K-1}{2}\right\rceil$ and its *counter* $(M(a))$ *is* $b = a_{\left\lfloor\frac{K+1}{2}\right\rfloor+i}$. Therefore, there are $i + \left(K - \left\lfloor\frac{K+1}{2}\right\rfloor - i\right) + 1 = K - \left\lfloor\frac{K+1}{2}\right\rfloor + 1$ candidates in $\{b \in A: b \leq a \; or \; M(a) \leq b\}$ which is at least $(K+1)/2$.
- If $M(a) < a$ then $a = a_i$ for some $i \geq \left\lfloor\frac{K+1}{2}\right\rfloor + 1$ and its *counter* $(M(a))$ *is* $b = a_{i-\left\lfloor\frac{K+1}{2}\right\rfloor}$. Therefore, there are at least $K - i + 1 + i - \left\lfloor\frac{K+1}{2}\right\rfloor = K - \left\lfloor\frac{K+1}{2}\right\rfloor + 1$ candidates in $\{b \in A: b \leq M(a) \; or \; a \leq b\}$ which is at least $(K+1)/2$.

In both cases we would have at least $(K+1)/2$ candidates in $\{b \in A: b \leq a \; or \; M(a) \leq b\}$ or in $\{b \in A: b \leq M(a) \; or \; a \leq b\}$. ■

F. Proof of Claim 4

We prove now **the first statement**. Given a profile π, denote the set of all candidates which is *best* in at least one of the preferences in $\pi(N)$ by*:* $B_\pi(A) = \{a \in A | \exists i \in N : \forall b \in A \backslash \{a\}, a \succ_i b\}$. Since π is single-caved profile, it must be that $B_\pi(A) \subseteq \{a_1, a_K\}$, and at least one of $\{a_1, a_K\}$ has *first place-frequency* (i.e., $\mu_\pi(\{\succ: rank_\succ(.) = 1\})$) of at least $n/2$, thus at least one of them must be (weak) Condorcet winner and therefore, at least one of them must be in $\mathbb{C}$, because otherwise $\mathbb{C}$ would be empty. Note that for $K = 3$ we obtain that $(K+1)/2 = 2$, but since $k < 2$, we have that either $\mathbb{C} = \{a_1\}$ or $\mathbb{C} = \{a_K\}$ depending on the *first place-frequency*, and the proof is trivial. So, assume now that $K > 3$. There are two cases for a_1:

Case 1: a_1 has *first place-frequency* of at least $n/2$. In this case, $a_1 \in \mathbb{C}$, because otherwise $\mathbb{C}$ would be empty. Now, Let $a_i \in \mathbb{C}$ (for some $2 < i < (K+1)/2$)) we will show that for any $1 < j < i, a_j \in \mathbb{C}$. Assume by contradiction that $a_j \notin \mathbb{C}$. Since $a_i \in \mathbb{C}$, it must be that $\mu_\pi\left(C\left(a_i \succ a_j\right)\right) > n/2$, but since π is single-caved, we obtain by Claim 3 that $[\forall \succ \in \pi(N): a_i \succ a_j \Rightarrow a_t \succ a_i, \forall t > i]$. Therefore, we obtain that $\mu_\pi\left(C(a_t \succ a_i)\right) \geq \mu_\pi\left(C\left(a_i \succ a_j\right)\right) > n/2$, and consequently it must be that also $a_t \in \mathbb{C}, \forall t > i$ (in addition to $a_1$and a_i). In this case we would have that: $|\mathbb{C}| \geq K - i + 1 + 1$. But since $i < (K+1)/2$ we would have that $|\mathbb{C}| \geq K - i + 1 + 1 > (K+1)/2 + 1$, which contradicts the assumption of the claim that $|\mathbb{C}| < (K+1)/2$.

Case 2: a_1 has *first place-frequency* less than $n/2$. Let $a_i \in \mathbb{C}$ (for some $1 < i < (K+1)/2$)), we will show first that $a_1 \in \mathbb{C}$. Assume by contradiction that $a_1 \notin \mathbb{C}$. Since $a_i \in \mathbb{C}$ then by Claim 3, it must be (for any single-caved preference relation) that $[a_i \succ a_1] \Rightarrow [a_t \succ a_i, \forall\, t > i]$, and consequently $\mu_\pi\left(C(a_t \succ a_i)\right) \geq \mu_\pi\left(C(a_i \succ a_1)\right)$. Since $a_i \in \mathbb{C}$, we obtain $\mu_\pi\left(C(a_i \succ a_1)\right) \geq n/2$.

Therefore, $\mu_\pi\big(C(a_t \succ a_i)\big) \geq \mu_\pi\big(C(a_i \succ a_1)\big) \geq n/2$. Consequently, it must be that $[a_t \in \mathbb{C}, \forall\, t > i\ \ or\ equivalently\ \forall\, a_i < a_t]$ (because otherwise, $a_i \notin \mathbb{C}$). In this case, $|\mathbb{C}| = K - i + 1$, but since $i < (K+1)/2$ we obtain that $|\mathbb{C}| > K - (K+1)/2 + 1 = (K+1)/2$, and consequently that $|\mathbb{C}| \geq (K+1)/2$ which contradicts the assumption of the claim that $|\mathbb{C}| < (K+1)/2$. We conclude that $a_1 \in \mathbb{C}$, but by the proof of case 1, it must be also that $a_j \in \mathbb{C}$ for every $a_1 < a_j < a_i$. By symmetry, the proof of **the second statement is similar** and is left to the reader.

We prove now **the third statement**. Recall that $|\mathbb{C}| < (K+1)/2$. Let $m = (K+1)/2$, assume by contradiction that $a_m \in \mathbb{C}$, and assume (w.l.g) that for some $i < (K+1)/2$, $a_i \notin \mathbb{C}$ (there must be such a candidate, since $k < K$). Now, since $\mathbb{C}$ is a Condorcet committee, we must have that $\mu_\pi\big(C(a_m \succ a_i)\big) > n/2$. Since $a_i < a_m < a_{m+1}$, then by Claim 3, it holds that $\mu_\pi(C(a_m \succ a_i) \leq \mu_\pi(C(a_{m+1} \succ a_m))$, but if $a_m \in \mathbb{C}$, we obtain that $\mu_\pi\big(C(a_{m+1} \succ a_m)\big) > n/2$, therefore a_{m+1} must be in $\mathbb{C}$ (otherwise a_m cannot be in $\mathbb{C}$), but then by the first statement (of Claim 4), it must be that $a_j \in \mathbb{C}, \forall j > (K+1)/2$, and consequently $|\mathbb{C}| \geq (K+1)/2$, which contradicts the assumption of Claim 4. ■

G. Probabilities for more than four candidates

Table 2: K = 4 and k = 1

n	$Pr_1 = Pr_2$	Pr_3
3	0	1
4	0	1
5	0.00783	0.99217
6	0	1
7	0.01419	0.98581
8	0	1
9	0.01869	0.98131
10	0.00025	0.99975
50	0.01415	0.98585
51	0.03392	0.96608
100	0.02186	0.97814
101	0.03327	0.96673
1000	0.03266	0.96734
1001	0.03378	0.96622
∞	0.03361	0.96639

Table 3: K = 4 and k = 2

n	Pr_1	Pr_2	Pr_3
3	0.11957	0	0.88043
4	0	0	1
5	0.17347	0.00030	0.82623
6	0.01415	0	0.98585
7	0.20216	0.00075	0.79710
8	0.03115	0	0.96885
9	0.21917	0.00113	0.77970
10	0.04771	0	0.95229
50	0.17993	0.00047	0.81960
51	0.27224	0.00333	0.72442
100	0.21798	0.00137	0.78065
101	0.26724	0.00301	0.72975
1000	0.26081	0.00288	0.73631
1001	0.26600	0.00295	0.73105
∞	0.26602	0.00291	0.73106

Table 4: K = 4 and k = 3

n	$Pr_1(2,1)$	$Pr_1(1,2)$	Pr_2	Pr_3
3	0.08191	0.80887	0	0.10922
4	0	0.48577	0	0.51423
5	0.11245	0.75552	0	0.13203
6	0.00505	0.53671	0	0.45824
7	0.12804	0.73195	0	0.14001
8	0.01242	0.57230	0	0.41528
9	0.13709	0.71926	0.00004	0.14361
10	0.02034	0.59732	0	0.38234
50	0.09416	0.67215	0.00003	0.23367
51	0.16386	0.68226	0.00040	0.15348
100	0.12362	0.68423	0.00021	0.19193
101	0.16292	0.68536	0.00044	0.15127
1000	0.15729	0.68762	0.00043	0.15467
1001	0.16099	0.68598	0.00037	0.15266
∞	0.16168	0.68693	0.00043	0.15097

Table 5: K = 5 and k = 1

n	Pr_1=Pr_2	Pr_3
3	0.00000	1.00000
4	0.00000	1.00000
5	0.00000	1.00000
6	0.00000	1.00000
7	0.00027	0.99973
8	0.00000	1.00000
9	0.00061	0.99939
10	0.00000	1.00000
50	0.00097	0.99903
51	0.00393	0.99607
100	0.00226	0.99774
101	0.00483	0.99517
1000	0.00593	0.99407
1001	0.00644	0.99356
1,000,000	0.00693	0.99307

Table 6: K = 5 and k = 2

n	Pr_1	Pr_2	Pr_3
3	0.00000	0.00000	1.00000
4	0.00000	0.00000	1.00000
5	0.01500	0.00000	0.98500
6	0.00000	0.00000	1.00000
7	0.02972	0.00000	0.97028
8	0.00268	0.00000	0.99732
9	0.03875	0.00000	0.96125
10	0.00707	0.00000	0.99293
50	0.05773	0.00005	0.94221
51	0.09454	0.00003	0.90542
100	0.07741	0.00002	0.92257
101	0.10435	0.00012	0.89553
1000	0.11112	0.00010	0.88878
1001	0.11881	0.00018	0.88100
1,000,000	0.11760	0.00025	0.88215

Table 7: K = 5 and k = 3

n	$Pr_1(2,1)$	$Pr_1(1,2)$	Pr_2	Pr_3
3	0.00000	0.20514	0.00000	0.79486
4	0.00000	0.09537	0.00000	0.90463
5	0.00268	0.27865	0.00000	0.71867
6	0.00024	0.14585	0.00000	0.85391
7	0.00603	0.31923	0.00000	0.67474
8	0.00035	0.19078	0.00000	0.80887
9	0.00815	0.34565	0.00000	0.64620
10	0.00168	0.22364	0.00000	0.77469
50	0.01675	0.38618	0.00000	0.59707
51	0.02740	0.44480	0.00000	0.52780
100	0.02255	0.43053	0.00000	0.54692
101	0.03235	0.46125	0.00000	0.50640
1000	0.03955	0.49097	0.00000	0.46948
1001	0.04147	0.49046	0.00000	0.46808
1,000,000	0.04114	0.50024	0.00002	0.45861

Table 8: K = 5 and k = 4

n	$Pr_1(3,1)$	$Pr_1(2,2)$	$Pr_1(1,3)$	Pr_2	Pr_3
3	0.00000	0.16053	0.66394	0.00000	0.17553
4	0.00000	0.11239	0.60736	0.00000	0.28025
5	0.00218	0.20187	0.62643	0.00000	0.16953
6	0.00064	0.14426	0.61695	0.00000	0.23816
7	0.00415	0.23010	0.61429	0.00000	0.15146
8	0.00150	0.16995	0.61660	0.00000	0.21195
9	0.00631	0.25127	0.60006	0.00000	0.14235
10	0.00284	0.18920	0.61445	0.00000	0.19351
50	0.01410	0.31594	0.56509	0.00000	0.10487
51	0.01959	0.34957	0.54357	0.00000	0.08727
100	0.01975	0.35186	0.54224	0.00000	0.08615
101	0.02425	0.37558	0.52341	0.00000	0.07676
1000	0.03247	0.41593	0.49410	0.00000	0.05751
1001	0.03412	0.41864	0.49063	0.00001	0.05661
1,000,000	0.03472	0.42264	0.48838	0.00001	0.05426

Table 9: K = 6 and k = 1

n	$Pr_1 = Pr_2$	Pr_3
3	0.00000	1.00000
4	0.00000	1.00000
5	0.00462	0.99538
6	0.00000	1.00000
7	0.00830	0.99170
8	0.00000	1.00000
9	0.01089	0.98911
10	0.00004	0.99996
50	0.00450	0.99550
51	0.01854	0.98146
100	0.00783	0.99217
101	0.01939	0.98061
1000	0.01635	0.98365
1001	0.01982	0.98018
1,000,000	0.01934	0.98066

Table 10: K = 6 and k = 2

n	Pr_1	Pr_2	Pr_3
3	0.02384	0.00000	0.97616
4	0.00000	0.00000	1.00000
5	0.05667	0.00013	0.94320
6	0.00102	0.00000	0.99898
7	0.07742	0.00017	0.92241
8	0.00280	0.00000	0.99720
9	0.09104	0.00023	0.90873
10	0.00578	0.00000	0.99422
50	0.05246	0.00005	0.94749
51	0.12083	0.00095	0.87822
100	0.06878	0.00009	0.93113
101	0.12763	0.00069	0.87168
1000	0.11349	0.00050	0.88600
1001	0.13166	0.00078	0.86757
1,000,000	0.12789	0.00080	0.87131

Table 11: K = 6 and k = 3

n	$Pr_1(2,1)$	$Pr_1(1,2)$	Pr_2	Pr_3
3	0.00200	0.22979	0.00000	0.76820
4	0.00000	0.01029	0.00000	0.98971
5	0.00832	0.30373	0.00000	0.68795
6	0.00000	0.03418	0.00000	0.96582
7	0.01313	0.33462	0.00000	0.65225
8	0.00000	0.05967	0.00000	0.94033
9	0.01834	0.35392	0.00004	0.62770
10	0.00016	0.08348	0.00000	0.91636
50	0.00766	0.24007	0.00000	0.75227
51	0.02858	0.38872	0.00009	0.58261
100	0.01037	0.29034	0.00007	0.69922
101	0.02716	0.39166	0.00005	0.58114
1000	0.02391	0.36670	0.00005	0.60934
1001	0.03061	0.39057	0.00005	0.57877
1,000,000	0.02757	0.39757	0.00005	0.57481

Table 12: K = 6 and k = 4

n	$Pr_1(3, 1)$	$Pr_1(2, 2)$	$Pr_1(1, 3)$	Pr_2	Pr_3
3	0.00000	0.14311	0.66658	0.00000	0.19031
4	0.00000	0.00212	0.27088	0.00000	0.72701
5	0.00330	0.17852	0.61550	0.00000	0.20268
6	0.00000	0.00756	0.29932	0.00000	0.69312
7	0.00468	0.20058	0.59610	0.00000	0.19864
8	0.00000	0.01417	0.36274	0.00000	0.62310
9	0.00715	0.20682	0.58826	0.00000	0.19776
10	0.00010	0.02556	0.39159	0.00000	0.58274
50	0.00190	0.11018	0.52586	0.00000	0.36205
51	0.01208	0.23483	0.55864	0.00000	0.19445
100	0.00354	0.14124	0.54350	0.00000	0.31172
101	0.01264	0.22867	0.56221	0.00000	0.19648
1000	0.00954	0.21475	0.55874	0.00000	0.21698
1001	0.01276	0.23507	0.55930	0.00003	0.19284
1,000,000	0.01268	0.23877	0.55625	0.00003	0.19227

Table 13: K = 6 and k = 5

n	$Pr_1(4, 1)$	$Pr_1(3, 2)$	$Pr_1(2, 3)$	$Pr_1(1, 4)$	Pr_2	Pr_3
3	0.00000	0.14877	0.63735	0.21387	0.00000	0.00000
4	0.00000	0.00151	0.16401	0.59587	0.00000	0.23861
5	0.00278	0.17358	0.58567	0.23285	0.00000	0.00513
6	0.00000	0.00587	0.21961	0.58455	0.00000	0.18998
7	0.00407	0.18277	0.56825	0.23752	0.00000	0.00739
8	0.00000	0.01210	0.26345	0.56828	0.00000	0.15617
9	0.00526	0.18789	0.56115	0.23735	0.00000	0.00836
10	0.00004	0.01924	0.29689	0.55042	0.00000	0.13341
50	0.00150	0.08975	0.46088	0.40063	0.00000	0.04724
51	0.00890	0.20079	0.53493	0.24307	0.00000	0.01231
100	0.00283	0.11542	0.49234	0.35566	0.00000	0.03376
101	0.00903	0.20088	0.53616	0.24131	0.00000	0.01262
1000	0.00727	0.17838	0.52640	0.27101	0.00000	0.01694
1001	0.00986	0.20193	0.53034	0.24429	0.00000	0.01358
1,000,000	0.00946	0.20426	0.52909	0.24379	0.00000	0.01340