Tax Evasion, Public Debt and Aggregate Instability (Very Preliminary Version)

Réda Marakbi, Patrick Villieu

University of Orléans, Laboratoire d'Économie d'Orléans, CNRS.

Abstract

In this article, we develop an endogenous growth model to analyze the relation between tax evasion and public debt accumulation. Our results are threefold. First, our model exhibits a multiplicity of equilibria in the long run: there is a low-growth and high public debt balanced growth path (BGP) and a high-growth and low-public debt BGP. Second, we show the existence of threshold effects in the tax evasion-public debt nexus. In low-growth economies, tax evasion negatively affects public debt while the relation between the two variables is characterized by a U-shaped curve in high-growth economies. Finally, regarding the local stability of the BGPs, we show that the high BGP is always well-determined. However, the topological behavior of the low BGP is more complex: it can either be locally determined, undetermined or overdetermined. In the latter case, a Hopf bifurcation appears depending on the level of tax evasion.

Keywords: Tax evasion, public debt accumulation, endogenous growth, multiple equilibria, Hopf bifurcation

1. Introduction

Tax evasion is one of the major public issues that most countries around the world are facing today. In 2013, the European Commission President Jose-Manuel Barroso said in a speech at the European Parliament that around 1 trillion euros is evaded annually in the EU member states. The United States are up against a similar problem. According to Rogoff (2017), tax evasion is estimated to be more than 3% of the US GDP every year. In Africa, a report of Global Financial Integrity (2013) has highlighted that tax evasion causes losses estimated at hundreds of millions of dollars per year. Thus, a number of governments, international organizations and NGOs have denounced in recent years the scale of tax evasion which is increasing steadily around the world. The recent revelations of the Panama Papers are an illustration of the worrisome increase of tax evasion between 1975 and 2016. In the same time, the stock of public debt held worldwide has increased from $26.85 trillion in 2005 to $56.89 trillion in 2016.[1] These observations raise a crucial question: what are the macroeconomic consequences of tax evasion in terms of public finance and public debt accumulation?

Surprisingly, to the best of our knowledge, there is no work that addresses the relation between tax evasion and public debt accumulation. Most of the macroeconomic literature on tax evasion has essentially focused on taxation policies.[2] Yet, Litina and Palivos (2013) have highlighted that tax evasion is a key part of the "Greek tragedy". Similarly, Pappa et al. (2015) show, in a DSGE model calibrated using Italian data, that tax evasion leads to substantial losses in output and welfare and amplifies the need to increase the tax rate in order to reduce the stock of public debt. They conclude that their results hold for countries like Greece, Spain and Portugal as well. Thus, tax evasion seems to strongly contribute to increase fiscal deficits in many European countries. Nevertheless, the relation may be more complex because tax evasion improves the efficiency of the private sector and may generate a complex interaction with economic growth. Strategic complementarities and multiple equilibria may emerge (Mauro, 2004; Aidt et al., 2008) and their implications in terms of public debt have not yet been explored in an endogenous growth setup.

The purpose of this paper is to fill this gap in the literature by developing a theoretical framework that allows assessing the relation between tax evasion and public debt accumulation. To this end, we build an endogenous growth model with productive public expenditures where public debt is introduced by relaxing the BBR assumption, in line with the experience of most countries that face positive deficit rates. Contrary to Minea and Villieu (2012) and Menuet et al. (2017) who have modeled similar mechanisms, we assume that households receive a risk premium that positively depends on tax evasion. In a first step, tax evasion is modeled by an exogenous fraction of the government's revenues that households evade in order to

[1] Source: https://www.statista.com/statistics/686067/global-public-debt/

[2] For example, Chen (2003) examines how tax evasion affects the optimal tax rate in an AK endogenous growth model with productive public expenditures and a balanced-budget rule (hereafter BBR). Similarly to Barro (1990) and Futagami et al. (1993), his model reproduces the inverted U-shaped curve between optimal taxation and growth. However, he shows that the optimal tax rate is higher as tax evasion becomes more widespread. According to Chen (2003), an increased tax rate allows compensating for the losses caused by tax evasion. However, this result holds only when the government has no other instruments to finance public spending. If we relax the BBR's assumption, the losses caused by tax evasion might also be financed by public borrowing.

increase their disposable income. As a second step, we endogenize tax evasion and consider the optimizing behavior of households who make an effort to evade as much taxes as possible.

Our results are threefold. First, in the steady-state, our model exhibits multiplicity of equilibria. Contrary to Mauro (2004), this multiplicity comes from the interaction between the intertemporal households' saving behavior and the government budget constraint. On the one hand, in the Keynes-Ramsey rule, economic growth is positively related to productive public expenditures that increase the real interest rate, and then, the public debt burden in the long run. On the other hand, productive public expenditures are positively linked to economic growth since growth allows reducing the public debt burden in the long run. This interaction generates a multiplicity of equilibria that leads to two steady-state solutions, i.e a high-growth and low-public debt solution and a low-growth and high-public debt solution.

Second, regarding the long run consequences of tax evasion on growth and public debt, we show that the low BGP positively depends on tax evasion while the effects of tax evasion on the high BGP are characterized by an inverted U-shaped relation. By contrast, the relation between tax evasion and public debt accumulation is negative in low-growth economies and characterized by a U-shaped curve in high growth economies. In high-growth economies, tax evasion exerts a dual effect on public debt. On the one hand, it increases the marginal productivity of private capital by improving the efficiency of the private sector leading to (i) a higher growth and (ii) a lower debt. On the other hand, tax evasion, by reducing tax revenues, (i) decreases growth and (ii) amplifies the need to resort to public borrowing to finance public spending.

Third, regarding the transition path, our model exhibits complex dynamics. While the high BGP is always saddle-path stable, the determinacy of the low BGP crucially depends on the elasticity of the risk premium (which is itself a function of tax evasion). The low BGP can either be locally determined (saddle-path stable), overdetermined (unstable) or undetermined (stable). In the latter case, a Hopf bifurcation occurs, leading to the emergence of limit-cycles and aggregate instability. We also show that the behavior of the government towards tax evasion is an essential variable for the determinacy of the low BGP.

The remainder of the paper is structured as follows. Section 2 presents the baseline model. In Section 3, we focus on the steady-state properties. Section 4 analyzes the dynamics of the model outside the steady-state. Section 5 proposes an extension of the model to the case where tax evasion is endogenous. Section 6 concludes.

2. The model

We consider a continuous-time endogenous growth model in a closed economy. The economy is populated by two perfectly rational agents: a private sector and a government.

2.1. The private sector

The private sector is represented by an infinitely-lived representative agent who produces and consumes a unique final good. His objective is to maximize the present value of the discounted sum of instantaneous utility functions based on consumption ($c_t > 0$).

$$\mathcal{U}(c_t) = \int_0^\infty \exp(-\rho t) u(c_t)\, \mathrm{d}t, \tag{1}$$

where $\rho \in (0, \infty)$ corresponds to the subjective discount rate.

To get an endogenous growth path in the long run, we assume a CES utility function where σ corresponds to the intertemporal elasticity of substitution

$$u(c_t) = \begin{cases} \dfrac{c_t^{1-\sigma} - 1}{1-\sigma} & \text{if } \sigma \neq 1, \\ \log(c_t) & \text{if } \sigma = 1. \end{cases} \tag{2}$$

Moreover, for $\mathcal{U}(c_t)$ to be bounded[3], we need to ensure that the no-Ponzi game constraint is satisfied, i.e. $(1-\sigma)\gamma_c < \rho$.[4]

The production function (y_t), based on physical capital (k_t) and productive public expenditures (g_t), is the same as in Barro (1990). It is described by the following relation :

$$y_t = A k_t^{1-\alpha} g_t^{\alpha}, \tag{3}$$

where A is a scale parameter and $0 < \alpha < 1$ is the elasticity of output to productive public expenditures (similarly, $0 < 1 - \alpha < 1$ is the elasticity of output to private

[3]This condition is necessary to find an optimum for welfare maximization.

[4]We note γ_x the growth rate of the variable x.

capital). All variables are per capita. For the sake of simplicity and without any loss of generality, population is normalized to unity.

In this model, households accumulate two assets: public debt securities b_t and private capital k_t. For simplicity, we abstract from capital depreciation. Therefore, the household's instantaneous budget constraint can be expressed as:[5]

$$\dot{k}_t + \dot{b}_t = [1 - \mathcal{P}(.)]\, R_t b_t + y_t^d - c_t. \tag{4}$$

Households use their income (y_t) to consume c_t, to accumulate capital k_t and save a part of their revenue in the form of government bonds (b_t). Government bonds are public debt securities and have a return rate noted R_t. Notice that the return rate of bonds (R_t) is different from the real interest rate (r_t). The latter is subject to a risk premium noted $\mathcal{P}(.)$, to be defined below. Finally, y_t^d corresponds to the disposable income of households expressed as

$$y_t^d = [1 - (1 - \eta)\tau] y_t, \tag{5}$$

where τ represents the flat-tax rate fixed by the government and η is a parameter denoting tax evasion. Indeed, we consider in this model that households may have incentives to evade taxes in order to increase their disposable income. As Huang and Wei (2006), Dimakou (2013) and Dimakou (2015) among others, we model tax evasion, in a first stage, by a simple exogenous parameter reducing tax revenues for the government. However, we also consider that this reduction of the tax revenues collected by the government leads to a proportional increase in the disposable income of households at the aggregate level. Thus, for $\eta \to 1$, tax evasion is widespread in the economy. Conversely, tax evasion is very low when $\eta \to 0$.

2.2. The government

The government determines the tax rate and borrows from the household in order to provide productive public expenditures. However, the government faces tax evasion. Following Huang and Wei (2006), Dimakou (2013), Dimakou (2015), we model tax evasion affects by a parameter η reducing tax revenues. Hence, the government budget constraint is expressed as

[5] A dot over a variable corresponds to the first derivative of this variable with respect to time: $\dot{x}_t := \partial x_t / \partial t$.

$$\dot{b}_t = R_t b_t + g_t - (1-\eta)\tau y_t. \tag{6}$$

The expression (6) constitutes an extension of the Barro's (1990) budget constraint for two reasons. *First*, it allows productive public expenditures to be financed by public borrowing. *Second*, the government faces a tax leakage because of by tax evasion.

In addition, we assume the risk premium to be a positive function of the level of tax evasion. The higher the level of tax evasion, the higher the risk premium. Thus, we define the expression of the risk premium as follows

$$\mathcal{P}(.) = 1 - \left\{ \frac{[1-\eta]\tau \bar{y}_t}{\bar{b}_t} \right\}^{\varepsilon}, \tag{7}$$

where ε denotes the sensitivity of the risk premium and $\bar{b}_t$ and $\bar{y}_t$ are respectively the equilibrium values of public debt and output. Notice that the representative household takes $\bar{b}_t$ and $\bar{y}_t$ as given values in his maximization program.

Finally, in order to obtain an endogenous growth solution, productive public expenditures must be endogenous in the government budget constraint and must therefore converge on some constant in the long run. To characterize this fact, we assume that the government follows the following fiscal rule

$$\dot{b}_t = \theta y_t, \tag{8}$$

where θ is a constant target of deficit.

2.3. Equilibrium

The equilibrium of the model is obtained by solving the household's program. This amounts to maximizing (1) subject to the constraints (2), (3), (4) and (5). The resolution of the household's maximization program is provided in Appendix A. It leads to the usual Keynes-Ramsey rule governing the law of motion of consumption

$$\gamma_c := \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\sigma}. \tag{9}$$

The goods market equilibrium provides the growth rate of capital

$$\gamma_k := \frac{\dot{k}_t}{k_t} = y_k - g_k - c_k. \tag{10}$$

where $g_k := g_t/k_t$ and $c_k := c_t/k_t$. The GDP-to-capital ratio is expressed as

$$y_k := \frac{y_t}{k_t} = A g_k^{\alpha}, \tag{11}$$

The law of motion of public debt stems from the fiscal rule followed by the government (8)

$$\gamma_b := \frac{\dot{b}_t}{b_t} = \frac{\theta y_k}{b_k}, \tag{12}$$

and b_k is obtained by using the government budget constraint.

$$\theta y_k = R_t b_k + g_k - (1-\eta)\tau y_k. \tag{13}$$

In addition, the tradeoff between public debt accumulation and private capital accumulation is given by the following relation

$$R_t = \frac{r_t}{1 - \mathcal{P}(.)}, \tag{14}$$

where $R_t > r_t \; \forall t$ and the real interest rate r_t is equal to the marginal productivity of capital such that $r_t = [1-(1-\eta)\tau](1-\alpha)y_k$. Notice that the return of bonds is positively related to the risk premium. The higher the risk premium, the higher the return of bonds.

Finally, replacing R_t by its expression in (14), we obtain the expression of the equilibrium public debt-to-capital ratio which only depends on the productive public expenditures-to-capital ratio

$$b_k = y_k \left\{ \frac{[(\theta + (1-\eta)\tau)y_k - g_k]\,[(1-\eta)\tau]^{\varepsilon}}{(1-\alpha)[1-(1-\eta)\tau]y_k^2} \right\}^{\frac{1}{1+\varepsilon}}. \tag{15}$$

3. The steady-state

3.1. The multiplicity of BGPs

We define a balanced growth path (hereafter BGP) where output, consumption, public debt, capital and productive public spending grow at a unique rate, namely γ^*[6] ($\gamma^* = \dot{y}_t/y_t = \dot{c}_t/c_t = \dot{b}_t/b_t = \dot{k}_t/k_t = \dot{g}_t/g_t$). Since we are interested in studying the relation between tax evasion, public debt and economic growth, we express the steady-state solution by two relations of b_k as functions of γ^*.

We get a first relation between b_k^* and γ^* from equation (12)

$$b_k^* = \frac{\theta A g_k^{*\alpha}}{\gamma^*} =: \mathcal{F}(\gamma^*), \tag{16}$$

where the steady-state ratio of productive public expenditures-to-capital is a function of γ^*. We obtain its expression by combining the Keynes-Ramsey rule ($\gamma^* = \sigma^{-1}(r^* - \rho)$) and the expression of the steady-state real interest rate ($r^* = (1-\alpha)(1-(1-\eta)\tau)Ag_k^{*\alpha}$)

$$g_k^* = \left[\frac{\sigma\gamma^* + \rho}{A[1-(1-\eta)\tau](1-\alpha)}\right]^{\frac{1}{\alpha}}. \tag{17}$$

From (15), we obtain a second relation between γ^* and b_k^*

$$b_k^* = y_k^* \left\{\frac{[(\theta + (1-\eta)\tau)y_k^* - g_k^*]\,[(1-\eta)\tau]^{\varepsilon}}{(1-\alpha)[1-(1-\eta)\tau]y_k^{*2}}\right\}^{\frac{1}{1+\varepsilon}} =: \mathcal{G}(\gamma^*), \tag{18}$$

where $y_k^* = Ag_k^{*\alpha}$.

Hence, the steady-state economic growth rate and the ratio of public debt-to capital are obtained at the intersection of equations (16) and (18).

Proposition 4.1. *(Multiplicity of BGPs) For small values of $\theta \geq 0$ and $\eta \geq 0$, there exists a non-empty set of parameters, denoted $\mathscr{C}$, which contains two and only two (positive) steady-state balanced-growth paths: a low BGP (noted γ^L) and a high BGP (noted γ^H), such that $0 < \gamma^L < \gamma^H$.*

[6] A star exponent (x^*) denotes the steady-state value.

Proof. We proceed in two steps. First, we show that there are two and only two BGPs for the special case $\theta = 0$. For $\dot{b}_t = b_t = 0$ and $\gamma > 0$, the ratio of public spending-to-capital is defined by

$$g_k^* = [A(1-\eta)\tau]^{\frac{1}{1-\alpha}} =: g_k^B, \tag{19}$$

and since the steady-state real interest rate is such that $r^B = (1-\alpha)[1-(1-\eta)\tau]A\left(g_k^B\right)^\alpha$, the associated economic growth rate is such that

$$\gamma^B = \sigma^{-1}\left[(1-\alpha)[1-(1-\eta)\tau]A\left(g_k^B\right)^\alpha - \rho\right]. \tag{20}$$

The couple $\left(\gamma^B, 0\right)$ denotes the "Barro solution" where $\gamma^B > 0$.

Moreover, we can obtain a no-growth solution, called the "Solow solution", for $\theta = \gamma = 0$. In this case, equation (16) is no longer defined but we can easily extract from (17) the corresponding expression of the productive public expenditures-to-capital ratio :

$$g_k^S = \left[\frac{\rho}{A(1-\alpha)(1-(1-\eta)\tau)}\right]^{\frac{1}{\alpha}}. \tag{21}$$

Therefore, b_k^S is a function of the parameters of the model

$$b_k^S = y_k^S\left\{\frac{\left[(1-\eta)\tau y_k^S - g_k^S\right][(1-\eta)\tau]^\varepsilon}{(1-\alpha)[1-(1-\eta)\tau]\left(y_k^S\right)^2}\right\}^{\frac{1}{1+\varepsilon}}. \tag{22}$$

Thus, the couple $\left(0, b_k^S\right)$ characterizes the second BGP of the model for $\theta = 0$.

The second step consists of generalizing the proof to the case where $\theta > 0$. In this case, it is clear that $\mathcal{F}(\gamma^*) \in C^\infty(\mathbb{R}^+)$ is a strictly decreasing and strictly convex function since $\mathcal{F}'(\gamma^*) = -\frac{\theta\rho}{(1-\alpha)[1-(1-\eta)\tau]\gamma^2} < 0$ and $\mathcal{F}''(\gamma^*) = \frac{2\theta\rho}{(1-\alpha)[1-(1-\eta)\tau]\gamma^3} > 0$. In addition, $\lim_{\gamma^*\to 0}\mathcal{F}(\gamma^*) = -\infty$ and $\lim_{\gamma^*\to+\infty}\mathcal{F}(\gamma^*) = 0$. On the other hand, we can observe that $\mathcal{G}(\gamma^*) \in C^\infty(]0, \bar{\gamma}^*[)$ is characterized by an inverted U-shaped curve on the interval $]0, \bar{\gamma}^*[$. Indeed, from $\mathcal{G}'(\gamma^*) = 0$, we can extract a threshold of growth (noted $\widehat{\gamma}^*$) which is equal to $\widehat{\gamma}^* = \frac{1}{\sigma}\left[(1-\alpha)[1-(1-\eta)\tau]A\left(\frac{\varepsilon\alpha[\theta+(1-\eta)\tau]A}{1-\alpha(1-\varepsilon)}\right)^{\frac{\alpha}{1-\alpha}} - \rho\right]$. Besides, it is quite obvious that $\mathcal{G}''(\gamma^*) < 0$. Therefore, the function $\mathcal{G}(\gamma^*)$ is concave. Finally, we have $\lim_{\gamma^*\to 0}\mathcal{G}(\gamma^*) = b_k^S$ and $\lim_{\gamma^*\to\bar{\gamma}^*}\mathcal{G}(\gamma^*) = 0$.

Since $\bar{\gamma}^* > \gamma^S$, according to the intermediate value theorem, there is a non-empty set of parameters $\mathscr{C}$ such that $\mathcal{F}(\gamma^*)$ and $\mathcal{G}(\gamma^*)$ intersect twice in the plane $\mathbb{R}^+ \times \mathbb{R}^+$ and give rise to two real and positive solutions for the steady-state growth rate noted γ_1^* and γ_2^*. Therefore, we define the low BGP as $\gamma^L \equiv \min(\gamma_1^*, \gamma_2^*)$ and the high BGP as $\gamma^H \equiv \max(\gamma_1^*, \gamma_2^*)$. $\square$

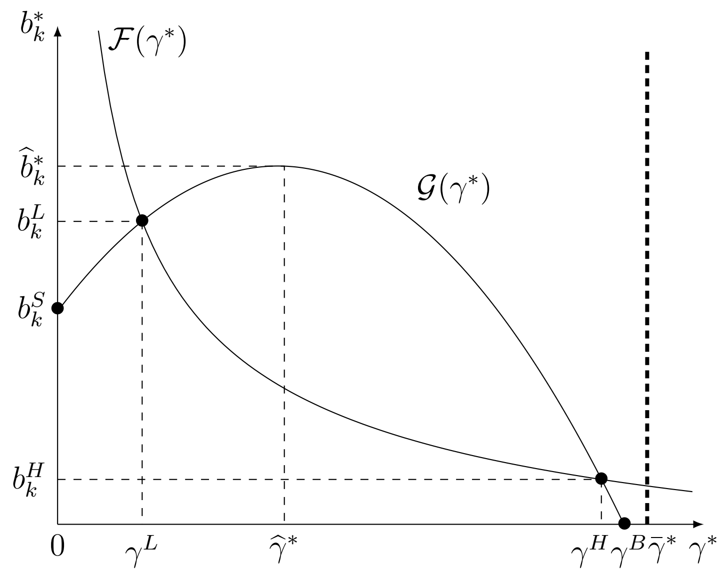

Figure 1: The steady-state

In our setup, the multiplicity of BGPs comes from the interaction between the Keynes-Ramsey rule and the government budget constraint. The intuitive explanation is the following. A low rate of economic growth amplifies the crowding-out effect on productive expenditures by increasing the debt burden which, in turn, further increases the public debt-to-capital ratio. Conversely, a high rate of economic growth, by increasing the real interest rate, increases the return of the government bonds which further reduces public debt accumulation. This interaction between economic growth and public debt (through productive public spending) generates multiplicity and leads to the emergence of two BGPs. Therefore, there are two equilibria in the long run: a low-growth and high-public debt equilibrium and a high-growth and low-public debt equilibrium.

In what follows, we will successively analyze the effects of tax evasion on growth and public debt accumulation in low-growth and high-growth economies, respectively.

3.2. The low steady-state

The low BGP is characterized by a low growth solution and a positive ratio of public debt-to-capital. Contrary to the Solow solution which defines and no growth solution ($\gamma^S = \theta = 0$), the low BGP gives rise to an economic growth rate slightly higher than zero. The Solow solution is reached for $\theta = 0$ while the low BGP is reached for $\theta > 0$ and $\gamma^L \to \gamma^S$ (with $\gamma^L > \gamma^S$).

Thus, from (6) and (18), we can approximate the expression of the low steady-state by the following two relations

$$\gamma^L \approx \frac{\theta}{b_y^S}, \tag{23}$$

where

$$b_y^S = \left\{ \frac{\left[(1-\eta)\tau - A^{-1}(g_k^S)^{1-\alpha}\right]\left[(1-\eta)\tau\right]^{\varepsilon}}{(1-\alpha)(1-(1-\eta)\tau)y_k^S} \right\}^{\frac{1}{1+\varepsilon}}. \tag{24}$$

Proposition 4.2 establishes the steady-state impact of tax evasion on economic growth in the neighborhood of the low BGP.

Proposition 4.2. *(Effects of tax evasion on growth and public debt in the neighborhood of the low BGP) In the neighborhood of the low BGP,*

(i) there is a negative relation between tax evasion and public debt accumulation.

(ii) tax evasion leads to an increase in economic growth.

Proof. See Appendix B.

In the neighborhood of the low BGP, any increase in tax evasion requires a lower debt and a higher growth to be compatible with the steady-state. The intuitive explanation of this result is the following. Tax evasion reduces tax revenues. Other things being equal, this leads to an increase in the risk premium, the return of the government bonds and consequently the debt burden in the long run. Thus, in the absence of another instrument of public finance (such as seigniorage for instance), the government has no choice but to decrease the deficit-to-output ratio to be able to pay down the debt. This reduction in public debt reduces the unproductive public expenditures related to the debt burden and allows the government to have more resources to finance productive public spending. Finally, since public debt and growth are negatively related in the long run, the negative impact of tax evasion on public debt implies a positive impact on economic growth.

3.3. The high steady-state

Proposition 4.3. *(Effects of tax evasion on growth and public debt in the neighborhood of the high BGP) In the neighborhood of the high BGP,*

(i) the relation between tax evasion and public debt accumulation is characterized by a U-shaped curve.

(ii) there is an inverted U-shaped relation between tax evasion and long-run growth.

Proof. See Appendix C

In the neighborhood of the high BGP, we find a critical level of tax evasion ($\widehat{\eta} \in (0, 1)$) such that the relation between tax evasion and growth is reversed. At low levels, the impact of tax evasion on growth is positive. At high levels, the impact of tax evasion on growth is negative. This is explained by the dual effect that tax evasion exerts on long-run growth. On the one hand, tax evasion positively affects long-run growth by stimulating capital accumulation and investment through the channel of the disposable income of households. On the other hand, tax evasion generates a reduction of the productive public expenditures provided by the government which has a detrimental impact on long-run growth. When the first effect dominates, the impact of tax evasion on growth is positive and conversely. As before, since public debt and growth are negatively related, the effects of tax evasion on growth and public debt are symmetric. Therefore, the threshold of tax evasion that maximizes growth minimizes the public debt ratio. At low levels, tax evasion reduces public debt (as in the neighborhood of the low BGP) but high levels of tax evasion lead to an increase in public debt in the long run.

4. Transitional dynamics

4.1. The reduced-form

To study the dynamics of the model outside the steady-state, we compute a two-variable reduced form (see Appendix D). The reduced-form of the model can either be expressed by two relations in c_k and g_k or in c_k and b_k. For convenience, we will study the topological behavior of the low BGP by resorting to the reduced-form in c_k and g_k while the topological behavior of the high BGP will be determined from the reduced-form in c_k and b_k.

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = \dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k, \\ \dfrac{\dot{g}_k}{g_k} = \dfrac{\mathcal{M}(g_k)}{g_k} \left[\dfrac{\theta A g_k^{\alpha}}{b_k} + c_k + g_k - y_k \right], \end{cases} \tag{25}$$

where

$$\mathcal{M}(g_k) = \frac{(1+\varepsilon)[(\theta + (1-\eta)\tau)y_k - g_k]g_k}{\varepsilon\alpha[(\theta + (1-\eta)\tau)y_k - g_k] - (1-\alpha)g_k}. \tag{26}$$

The first relation of system (25) corresponds to the Keynes-Ramsey rule that characterizes the consumption behavior while the second relation establishes the law of motion of productive public expenditures. As previously mentioned, we can also express the second relation as the law of motion of public debt where productive public spending is a function of public debt.

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = \dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k, \\ \dfrac{\dot{b}_k}{b_k} = \dfrac{\theta A g_k^{\alpha}}{b_k} + c_k + g_k - y_k. \end{cases} \tag{27}$$

In both reduced-forms, there is one jump variable (the ratio of consumption-to-capital c_k) and one predetermined variable (the ratio of productive public expenditures-to-capital g_k or the ratio of public debt-to capital b_k). The ratio of productive public expenditures-to-capital is a predetermined variable since g_k depends on b_k and b_k can never jump because b_t and k_t are predetermined $\forall t$.

In order to study the local stability of the BGPs, we resort to linearized forms of systems (25) and (27) in the neighborhood of BGP i ($i \in \{L, H\}$).

$$\begin{pmatrix} \dot{c}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}^i \begin{pmatrix} c_k - c_k^{*i} \\ g_k - g_k^{*i} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \dot{c}_k \\ \dot{b}_k \end{pmatrix} = \mathbf{J}^i \begin{pmatrix} c_k - c_k^{*i} \\ b_k - b_k^{*i} \end{pmatrix} \tag{28}$$

where $\mathbf{J}^i$ is the Jacobian matrix in the neighborhood of BGP i. Since there is one predetermined variable and one jump variable, the Blanchard-Kahn conditions are fulfilled if and only if the Jacobian matrix contains two opposite-signs eigenvalues.

Thus, to analyze the local determinacy or indeterminacy of the BGPs, we study the following characteristic polynomial of degree 2 associated with the steady-state i

$$\mathscr{P}^i(\lambda) = \lambda^2 - \mathcal{T}(\mathbf{J}^i)\lambda + \mathcal{D}(\mathbf{J}^i) = 0, \tag{29}$$

where the roots of the characteristic polynomial (λ_1 and λ_2) correspond to the eigenvalues of the Jacobian matrix and $\mathcal{T}(\mathbf{J}^i)$ and $\mathcal{D}(\mathbf{J}^i)$ are respectively the trace and the determinant of the Jacobian matrix in the neighborhood of the steady-state i. In order to study the topological behavior of each steady-state, the following proposition successively establishes the determinant and the trace of the high and the low BGPs for low values of the deficit target[7].

Proposition 4.4. *(Determinant and trace of the Jacobian matrix in the neighborhood of BGP i). For low values of θ (i.e. $\theta \to 0$), the determinant and the trace of the Jacobian matrix in the neighborhood of BGP i are the following.*

(i) In the neighborhood of the Barro BGP:

$$\mathcal{D}(\mathbf{J}^B) = -\gamma^B c_k^B$$
$$\mathcal{T}(\mathbf{J}^B) = -\gamma^B + c_k^B$$

(ii) In the neighborhood of the Solow BGP:

$$\mathcal{D}(\mathbf{J}^S) = -\tilde{\mathcal{M}}\left(g_k^S\right) c_k^S (1-\alpha)[1-(1-\eta)\tau]\sigma^{-1} A \left(g_k^S\right)^{\alpha-1}$$
$$\mathcal{T}(\mathbf{J}^S) = c_k^S + \tilde{\mathcal{M}}\left(g_k^S\right)\left[1 - \alpha A \left(g_k^S\right)^{\alpha-1}\right]$$

where $\tilde{\mathcal{M}}(g_k^S) = \mathcal{M}(g_k^S)\Big|_{\theta\to 0}$.

Proof. See Appendix E.

4.2. Local stability of the Barro BGP

From Proposition 4.4, we can directly establish the determinacy of the Barro BGP as follows.

[7] In order to obtain an endogenous growth solution, the deficit ratio must be sufficiently low.

Proposition 4.5. *(Determinacy of the Barro BGP) The Barro BGP is locally determined (i.e. saddle-path stable) for any values of the parameters.*

Proof. From Proposition 4.4, it is clear that the determinant of the Barro BGP is always negative. Therefore, $\mathscr{P}^B(0) = -\gamma^B c_k^B < 0$ and the Jacobian matrix in the neighborhood of the Barro BGP contains two opposite-signs real eigenvalues. This ensures the fulfillment of the Blanchard-Kahn conditions and, therefore, the local determinacy of the Barro BGP. □

4.3. Local stability of the Solow BGP

The determinacy of the Solow BGP is more complicated to establish since it crucially depends on the sign of $\tilde{\mathcal{M}}(g_k^S)$ and, consequently, on the value of the elasticity of the risk premium. Specifically, three configurations are possible. First, if $\mathcal{D}(\mathbf{J}^S) < 0$, the Jacobian matrix will possess two opposite-signs eigenvalues and the Solow BGP will be saddle-path stable. Second, if $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) > 0$, the Jacobian matrix will contain two eigenvalues with positive real parts and the BGP will be overdetermined (unstable). This configuration, which corresponds for instance to the case where $\varepsilon \to 0$ and $\eta \to 0$, has extensively been studied by Minea and Villieu (2012). Finally, if $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) < 0$, the two eigenvalues of the Jacobian matrix will have negative real parts and the Solow BGP would be undetermined (stable). The transition between the second case and the third case is featured by a Hopf bifurcation. At that point, the Jacobian matrix contains a pair of complex conjugate eigenvalues, with real parts equal to zero.

In what follows, in order to characterize the local dynamics of the Solow BGP, we will consider the elasticity of the risk premium as our bifurcation parameter[8]. In a model close to ours but without risk premium and tax evasion (i.e. $\varepsilon = \eta = 0$), Minea and Villieu (2012) have indeed shown that the Solow BGP is always unstable. Therefore, we will analyze the values of ε at which $\tilde{\mathcal{M}}(g_k^S)$, and then the determinant and the trace of the Jacobian matrix, change sign in the neighborhood of the Solow BGP.

Lemma 4.1. *The expression of the value of ε at which $\tilde{\mathcal{M}}(g_k^S)$ changes sign (noted $\tilde{\varepsilon}$) is given by*

[8]We could also consider the tax evasion parameter to characterize the value at which a Hopf bifurcation appears. However, in that case, is is more complicated to propose analytical proofs as the expression of the bifurcation parameter would be given by an implicit function.

$$\tilde{\varepsilon} = \frac{(1-\alpha)g_k^S}{\alpha(1-\eta)\tau y_k^S - \alpha g_k^S}.$$

Proof. In order for the stock of public debt to be positive, the numerator of $\tilde{\mathcal{M}}(g_k^S)$ must be positive. Therefore, we obtain the expression of $\tilde{\varepsilon}$ by equalizing the denominator of $\tilde{\mathcal{M}}(g_k^S)$ to zero ($\varepsilon\alpha[(1-\eta)\tau y_k^S - g_k^S] - (1-\alpha)g_k^S = 0$). □

Lemma 4.2. *The Hopf bifurcation occurs at the unique point ε^h defined as*

$$\varepsilon^h = \frac{(1-\alpha)g_k^S c_k^S + (\alpha y_k^S - g_k^S)[(1-\eta)\tau y_k^S - g_k^S]}{[(1-\eta)\tau y_k^S - g_k^S][g_k^S - \alpha(y_k^S - c_k^S)]}.$$

Proof. The expression of the value of the elasticity of the risk premium at which a Hopf bifurcation occurs stems from the equalization of the trace of the Jacobian matrix in the neighborhood of the Solow BGP to zero ($\mathcal{T}(\mathbf{J}^S) = 0$). □

Proposition 4.6. *(Determinacy of the Solow BGP) The topological properties of the Solow BGP can be summarized as follows :*

(i) if $\varepsilon > \tilde{\varepsilon}$, the Solow BGP is saddle-path stable.

(ii) if $\varepsilon < \tilde{\varepsilon}$ and $\varepsilon = \varepsilon^h$ (where $\varepsilon^h < \tilde{\varepsilon}$), a Hopf bifurcation occurs in the neighborhood of the Solow BGP. Thus, when $\varepsilon < \varepsilon^h$, the Solow BGP is overdetermined (unstable) and when $\varepsilon > \varepsilon^h$, the Solow BGP becomes undetermined (stable).

Proof. Let us successively consider the case $\varepsilon > \tilde{\varepsilon}$ and the case $\varepsilon < \tilde{\varepsilon}$.

(i) If $\varepsilon > \tilde{\varepsilon}$, then $\tilde{\mathcal{M}}(g_k^S) > 0$ and $\mathcal{D}(\mathbf{J}^S) < 0$. In that case, the Jacobian matrix in the neighborhood of the Solow BGP contains two opposite signs eigenvalues, namely $\lambda_{1,2} = \frac{1}{2}\left[\mathcal{T}(\mathbf{J}^S) \pm \sqrt{\mathcal{T}(\mathbf{J}^S)^2 - 4\mathcal{D}(\mathbf{J}^S)}\right]$. Hence, the Solow BGP is therefore saddle-path stable.

(ii) If $\varepsilon < \tilde{\varepsilon}$, then $\tilde{\mathcal{M}}(g_k^S) < 0$ and three different cases are possible. First, if $\varepsilon = \varepsilon^h$ where $\varepsilon^h < \tilde{\varepsilon}$, we have $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) = 0$. In this case, the Jacobian matrix contains two complex conjugate eigenvalues $\left(\lambda_{1,2} = \pm i\sqrt{\mathcal{D}(\mathbf{J}^S)}\right)$ and a Hopf bifurcation occurs. Second, if $\varepsilon^h < \varepsilon < \tilde{\varepsilon}$, then the determinant of the Jacobian matrix in the neighborhood of the Solow BGP is positive while its trace is negative, meaning that both eigenvalues have negative real parts. In this case, the Solow BGP is stable. Third, if $\varepsilon < \varepsilon^h < \tilde{\varepsilon}$, then both the determinant and the trace of the Jacobian matrix in the neighborhood of the Solow BGP are positive, meaning that both eigenvalues have positive real parts. Therefore, the Solow BGP is overdetermined (unstable).

$\square$

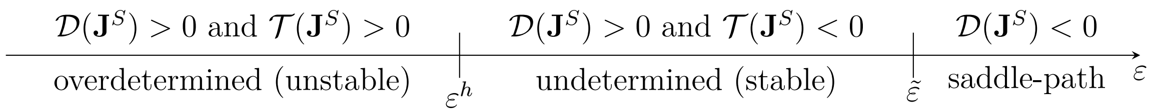

Figure 2: Local stability of the Solow BGP depending on the parameter ε

Table 1 proposes a numerical illustration showing the existence of Hopf bifurcations occuring at the Solow BGP for a constellation of parameters. Our baseline calibration is based on reasonable values of parameters. The discount rate is fixed at $\rho = 0.1$ and the risk-aversion coefficient is $\sigma = 1$. Regarding the technology, the total factor productivity parameter is set at $A = 0.6$ and the share of productive public spending is $\alpha = 0.35$ in order to obtain a capital share (0.715) close to that of Gomme et al. (2011). Following Trabandt and Uhlig (2011) and Gomes et al. (2013), the income tax rate is fixed at $\tau = 0.5$. In order to obtain a realistic economic growth rate, we set $\eta = 0.18$. Initially, we consider $\theta = 0$ to illustrate the zero-growth case. Thereafter, we will analyze the case where $\theta > 0$ to go beyond Proposition 4.6 and numerically explore the existence of Hopf bifurcations in the neighborhood of the low BGP (and not only in the neighborhood of the Solow BGP). Table 1 provides several numerical simulations showing the robustness of the Hopf bifurcation's occurence in the neighborhood of the Solow and the low BGPs for numerous sets of parameters.

Table 1 shows that aggregate instability emerges at low levels of the elasticity of the risk premium (when $\varepsilon < \varepsilon^h$). Besides, we can observe that the value of the elasticity of the risk premium at which the dynamics of the model in the neighborhood

θ	0			0.005	0.01	0.02
η	0.175	0.18	0.185	0.18	0.18	0.18
τ	0.5	0.5	0.5	0.5	0.48	0.52
ε^h	11.6287	11.6267	11.6312	13.1382	15.2915	19.8982
g_k^{*L}	0.0936	0.0925	0.0913	0.0957	0.0918	0.01108
c_k^{*L}	0.1682	0.1683	0.1683	0.1670	0.1658	0.1634
b_y^{*L}	0.4221	0.4199	0.4177	0.4122	0.3964	0.4194
γ^{*L}	0	0	0	0.0012	0.0025	0.0036
Lyap. coef.	-1.415e+05	-3.672e+05	-2.3579e+06	-4.8848e+04	-3.1733e+05	-7.9899e+03

Table 1: Bifurcation points and Lyapunov coefficients for a constellation of parameters

of both the Solow and the low BGP dramatically changes is higher as the level of tax evasion and the deficit ratio increase. Therefore, the elasticity of the risk premium must be higher when tax evasion and the deficit increase in order to prevent low growth-economies from falling into an indeterminate equilibrium.

The last row of Table 1 also provides an important information about the nature of the Hopf bifurcation. Simulations, run with ©matcont and presented in Table 3, show that the so-called first Lyapunov coefficient is always negative for numerous sets of parameters. This means that the Hopf bifurcations are supercritical and lead to the emergence of stable limit-cycles. Therefore, orbitally stable limit cycles occur for both the the Solow BGP and the low BGP with positive deficit ratio. Figure 3 depicts the family of limit-cycles bifurcating from the Hopf point for our baseline calibration.

5. The model with endogenous tax evasion

In this section, we extend the previous framework to the case where tax evasion is endogenous. To this end, we now consider that households make an effort e_t to evade taxes such that $e_t \in (1, \infty)$ and the government invests resources noted h_t to fight against tax evasion. Both e_t and h_t are endogenous variables. Thus, households no longer evade $\eta\tau y_t$ but a portion of taxes $\eta(e_t)\tau y_t$ where

$$\eta(e_t) = \eta_0 \Gamma(e_t), \tag{30}$$

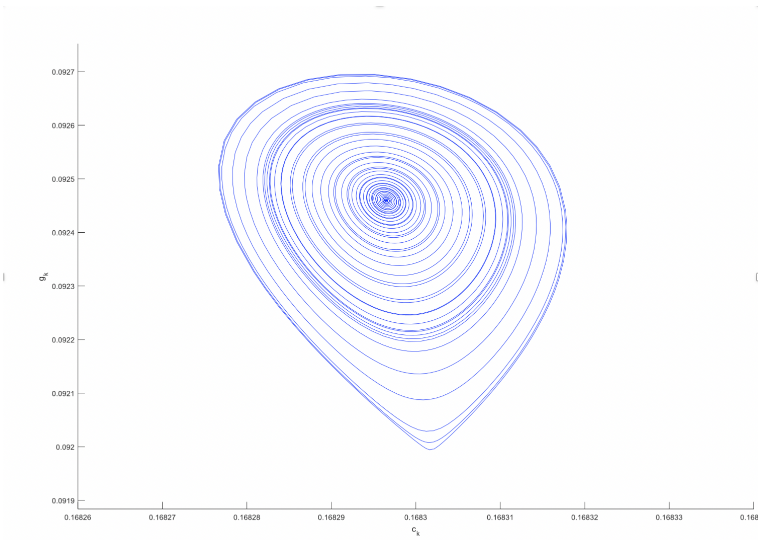

Figure 3: Family of limit-cycles bifurcating from the Hopf point

where η_0 corresponds to the initial level of tax evasion and $\Gamma(e_t)$ is an increasing and concave function ($\Gamma'(e_t) > 0$ and $\Gamma''(e_t) < 0$). In addition, $\eta(1) = \eta_0$ and $\lim\limits_{e_t \to \infty} = 1/\eta_0$ so that $\eta(e_t) \in (\eta_0, 1)$. To satisfy these properties, we assume that $\Gamma(e_t) := e_t^{\beta}$ where $\beta \in (0, 1)$.

In addition, in order to fight against tax evasion, the government devotes ressources noted h_t that evolve according to the following dynamics

$$\dot{h}_t = \xi \left[\eta(e_t) - \bar{\eta}\right] h_t, \tag{31}$$

where $\bar{\eta}$ represents the level of tax evasion targeted by the government and ξ is a strictly positive parameter denoting the speed of adjustment between the current and the targeted levels of tax evasion. The target $\bar{\eta}$ can also correspond to the requirements of the international organizations that may impose to the policymakers to improve their fiscal discipline to grant loans.

Henceforth, the government faces the following budget constraint

$$\dot{b}_t = R_t b_t + g_t - [1 - \eta(e_t)]\tau y_t - h_t e_t, \tag{32}$$

where the expression of R_t is given by (14) and the real interest rate is similar to that of the model with exogenous tax evasion: $r_t = (1 - \alpha)[1 - (1 - \eta(e_t))\tau]y_k$.

For households, h_t contributes to increase the cost of tax evasion. Therefore, the new budget constraint of the representative household is expressed as follows

$$\dot{k}_t + \dot{b}_t = [1 - \mathcal{P}(.)]\, R_t b_t + [1 - (1 - \eta(e_t))\tau] y_t - c_t - h_t e_t, \tag{33}$$

where h_t denotes the unit cost of the effort to evade taxes and $h_t e_t$ is the total cost.

Thus, from the first order condition of the hamiltonian with respect to e_t, we obtain the level of efforts chosen by households e_t to evade taxes

$$e_t = \left(\frac{\eta_0 \beta \tau y_k}{h_k} \right)^{\frac{1}{1-\beta}}. \tag{34}$$

In equation (34), we can notably observe that contractionary fiscal policies lead to more tax evasion, in line with Allingham and Sandmo (1972) and Cerqueti and Coppier (2011).

In what follows, we will successively study the properties of the model with endogenous tax evasion in the steady-state and the transitional dynamics in the neighborhood of the two BGPs.

5.1. The long-run solution

As in the exogenous tax evasion case, the steady-state solution is obtained at the intersection between two relations derived from the Keynes-Ramsey rule, the deficit rule and the government budget constraint (see Appendix F). The steady-state is overall similar to the case where tax evasion is exogenous but a new relation describing the long run tax evasion, that positively depends on long run growth and the level of tax evasion targeted by the government, appears

$$\eta(e^*) = \frac{\gamma^*}{\xi} + \bar{\eta}. \tag{35}$$

When tax evasion is endogenous, it is however rather difficult to extract analytical expressions for the long run economic growth rate and the long run public debt-to-capital ratio, even in the case where $\theta \to 0$. Consequently, we resort to numerical simulations (see Figure 4) and show that the steady-state solution is similar to that of the exogenous tax evasion case. As previously, when tax evasion is endogenous, the model is characterized by a low-growth and high-public debt solution and a high-growth and low-public debt solution.

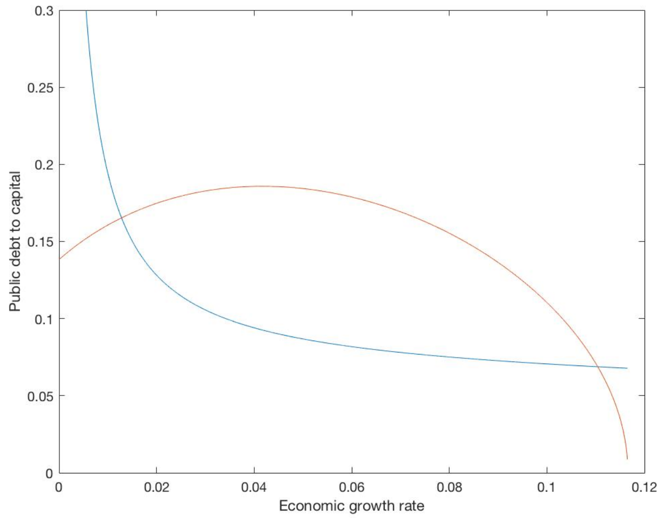

Figure 4: The steady-state solution

Further on, we study the long run effects of a change in the level of tax evasion targeted by the government and the elasticity of the risk premium. Figure 5 and Figure 6 respectively summarize the impact of an increase in the tax evasion target and the deficit target on public debt, growth and tax evasion in the neighborhood of both BGPs. In addition, in order to obtain normative results, we study the effects of a change in the tax evasion target and the deficit target on long run welfare.

In the neighborhood of the low BGP, any increase in the targeted level of tax evasions leads to an increase in growth, welfare and tax evasion and a decrease in public debt. In low-growth economies, one can observe that $\eta(e^*) \approx \bar{\eta}$, thereby explaining the positive impact of an increase in the tax evasion target on the degree of tax evasion in the economy. This positive relation between the targeted level of tax evasion and tax evasion generates an increase in the debt burden by increasing the risk premium, which, in turn, decreases public debt and increases long run growth and long run welfare (since public debt and growth are negatively related). In the neighborhood of the high BGP, positive variations of the tax evasion target give rise to threshold effects. The relation between the tax evasion target and growth, welfare and tax evasion is characterized by an inverted U-shaped curve while there is a U-shaped relation between the tax evasion target and the public debt ratio. The rationale behind these nonlinearities is the following. A higher target of tax evasion (i) allows reducing public debt and consequently enhancing growth and welfare and

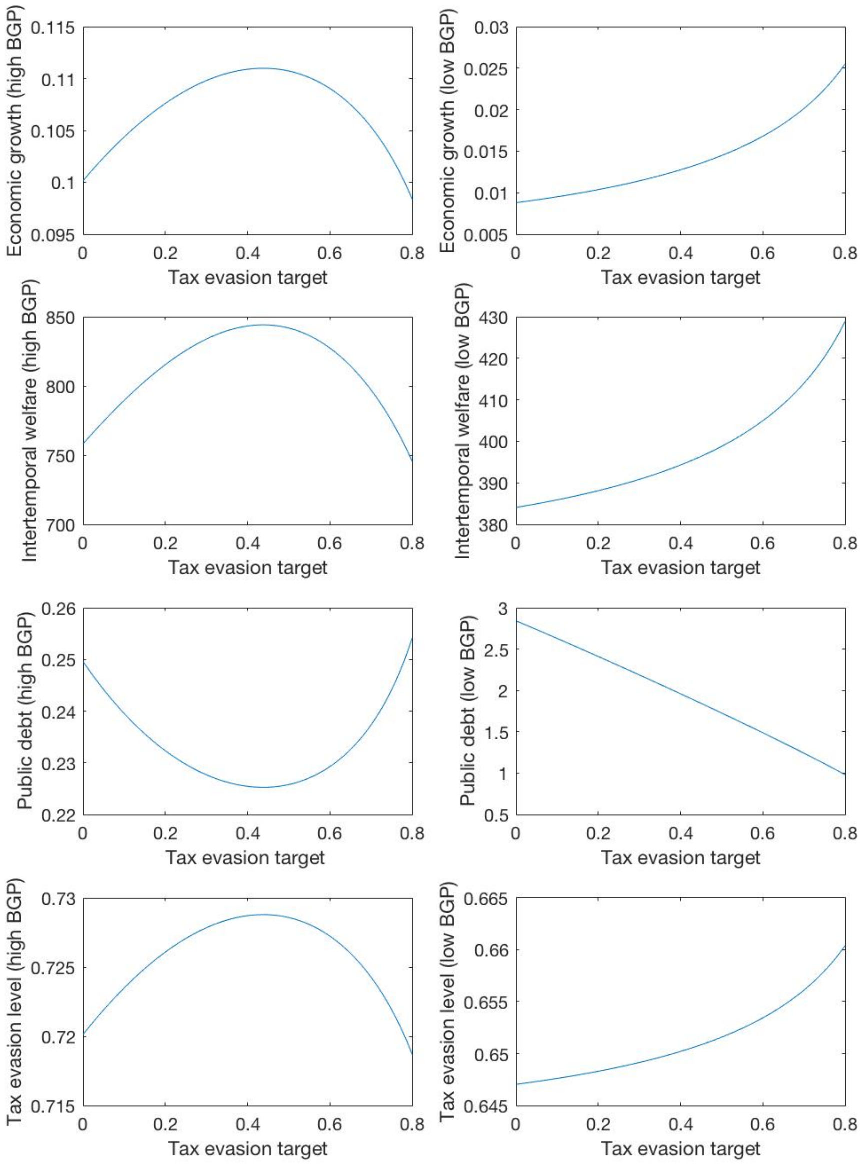

Figure 5: Effects of a change in $\bar{\eta}$

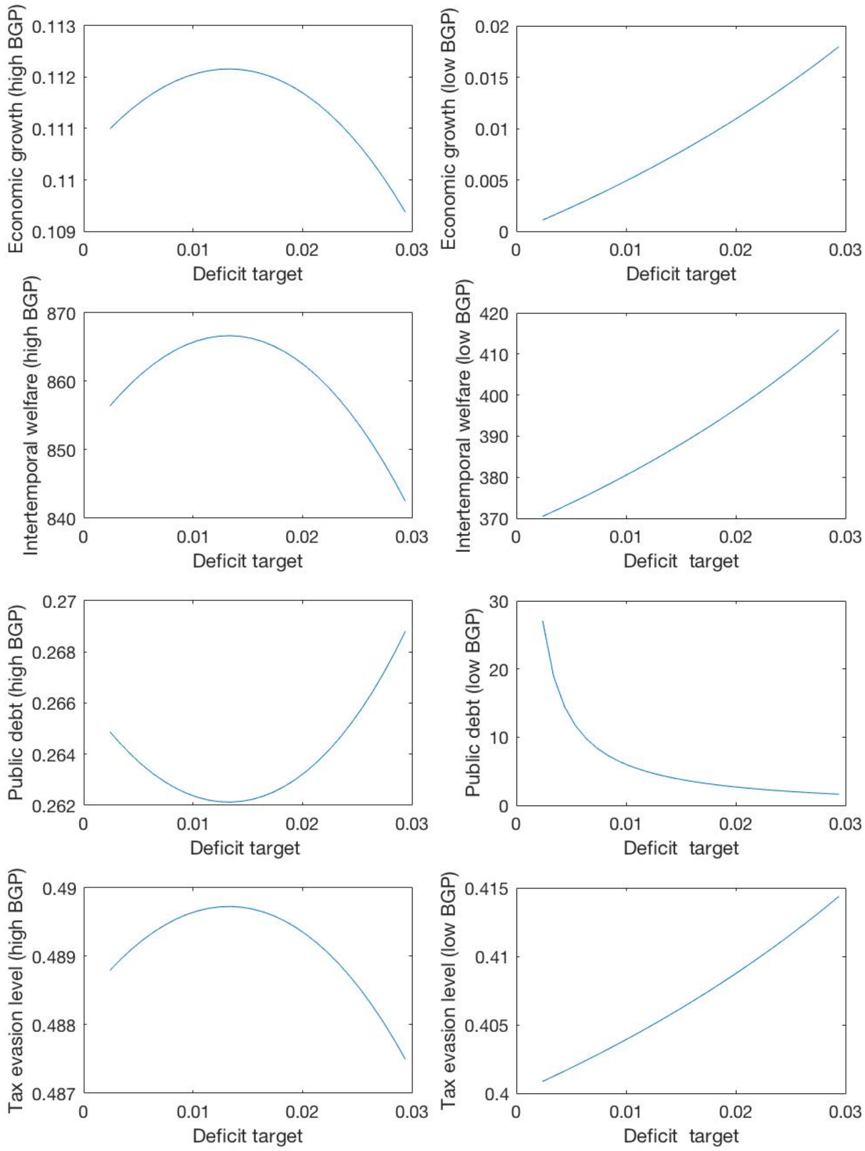

Figure 6: Effects of a change in θ

(ii) reduces the productivity of public expenditures leading to a decrease in growth and welfare and an increase in public debt. Since tax evasion depends on economic growth in the long run, there is a threshold effect in the tax evasion target-tax evasion nexus as well.

In Figure 6, we explore the impact of a change in the deficit ratio on growth, welfare, public debt and tax evasion in the long run. Alongside the high BGP, the effect of deficit on growth is double. It increases the resources available to finance public expenditures (positive effect) and increases unproductive public spending by rising the debt burden (negative effect). These two contradictory effects result in an inverted U-shaped relation between the deficit ratio and long run growth and, since they are positive functions of growth, between the deficit ratio and long run welfare and long run tax evasion. Symmetrically, an increase in θ gives rise to a U-shaped curve between the deficit ratio and public debt accumulation. Along the low BGP, the increase in the deficit ratio always reduces the debt burden, and then public debt, and therefore improves growth and welfare and rises tax evasion in the steady-state.

The presence of threshold effects between the tax evasion target and growth, welfare and public debt on the one hand, and between the deficit ratio and growth, welfare and public debt on the other, suggests the existence of interior solutions for an optimal policy mix composed of the tax evasion target and the deficit ratio in the neighborhood of the high BGP. Figures 7 and 8 derive the optimal policy mix in terms of growth and welfare, respectively, while Figure 9 depicts the policy mix that minimizes public debt. Overall, we can observe that the optimal solution is roughly the same. Based on our baseline calibration, the government should fix the deficit target approximatively at 1.3% and the tax evasion target at around 50% in order to maximize economic growth and intertemporal welfare and to minimize public debt.

5.2. Determinacy

Outside the steady-state, the model with endogenous tax evasion can be summarized by a three-variable reduced form in c_k, b_k and h_k

$$\begin{cases} \dot{c}_k = \left(\dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k\right) c_k, \\ \dot{h}_k = \left[\xi\left(\eta(e_t) - \bar{\eta}\right) + c_k + g_k - y_k\right] h_k, \\ \dot{g}_k = \mathcal{K}(.) \left\{ (1+\varepsilon)\left[\dfrac{\theta y_k}{b_k} + c_k + g_k - y_k\right] - x(.)\dfrac{\dot{h}_k}{h_k} \right\}, \end{cases} \tag{36}$$

where the expression of the constant $\mathcal{K}(.)$ and the function $x(.) \equiv x(g_k, h_k)$ are given in Appendix G.

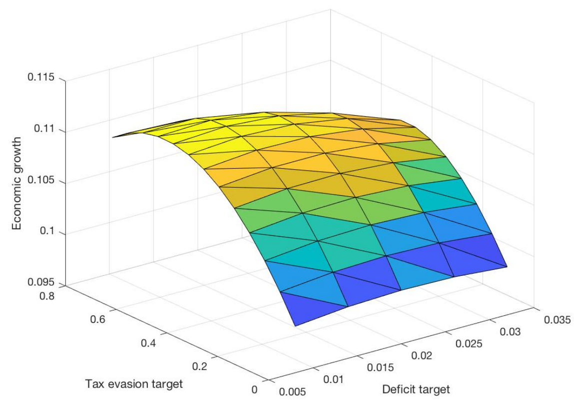

Figure 7: The growth-maximizing policy mix (high BGP)

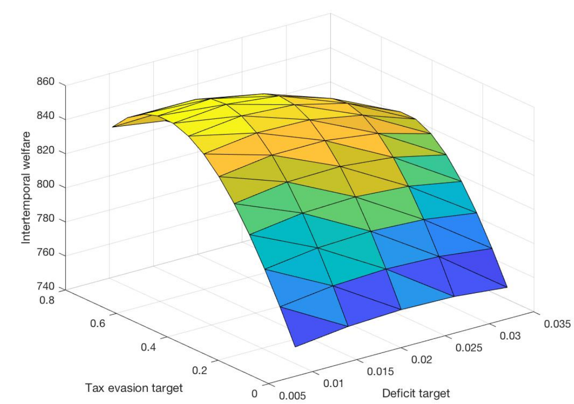

Figure 8: The welfare-maximizing policy mix (high BGP)

The system (36) is composed of one jump variable (c_k) and two predetermined variables (h_k and g_k). Therefore, the BGP i ($i \in (L, H)$) is locally determined if and only if the Jacobian matrix associated with a linearized form of the system (36) contains two negative eigenvalues and one positive eigenvalue. In a three-dimensional system, a Hopf bifurcation can appear if the Jacobian matrix contains one negative eigenvalue (with no imaginary part) and a pair of conjugate complex eigenvalues (with no real part). Thus, when the Jobian matrix is characterized by three negative eigenvalues, the economy is stable (overdeterminacy of the BGP) while two positive eigenvalues and one negative eigenvalue generate aggregate instability (indeterminacy

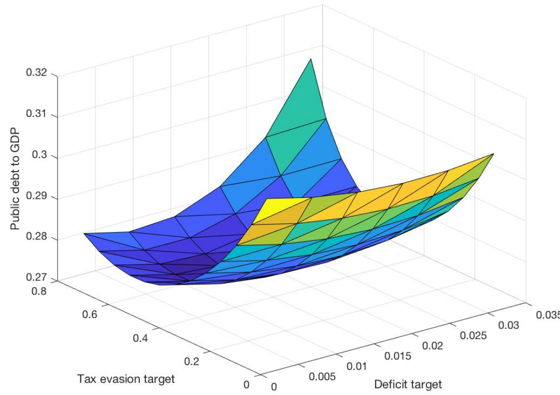

Figure 9: The public debt-minimizing policy mix (high BGP)

of the BGP).

Since it is rather difficult to derive analytical results when tax evasion is endogenously determined, we use numerical simulations to examine the local stability of both BGPs[9]. Our simulations highlight that the Jacobian matrix in the neighborhood of the high BGP always contains two negative and one positive eigenvalues. Therefore, since there are two jump variables and one predetermined variable in the system (36), the Blanchard-Kahn conditions are fulfilled and the high BGP is always locally stable. Table 2 reports the eigenvalues of the Jacobian matrix associated with system (36) in the neighborhood of the high BGP.

However, in the neighborhood of the low BGP, Figure 10 shows that aggregate instability may emerge under some conditions. Specifically, depending on the level of tax evasion targeted by the government ($\bar{\eta}$) and the speed of adjustment of the current level of tax evasion to this target (ξ), the low BGP might be undetermined (stable) or overdetermined (unstable). A Hopf bifurcation emerges at the point where the system switches from stability to instability. Thus, the "frontier stability" depicts the different pairs of values (ξ, $\bar{\eta}$) such that Hopf bifurcations occur. We can observe that stability nonlinearly depends on the speed of adjustment ξ and the tax evasion target $\bar{\eta}$. More precisely, aggregate instability appears below the stability frontier. Above the stability frontier, the low BGP is stable. Therefore, the emergence of aggregate instability in the neighborhood of the low BGP depends on the behavior of the government towards tax evasion. Specifically, when the government does not

[9]The values of parameters are similar to those of the exogenous tax evasion case.

$\bar{\eta}$ \ ξ	1	5	10	50
0.05	0.1039 -0.0145 -0.0540	0.1101 -0.0135 -0.1315	0.1109 -0.0133 -0.2257	0.1116 -0.0132 -0.9781
0.30	0.0948 -0.1059 -0.0110	-0.4037 0.0996 -0.0121	-0.7734 0.1002 -0.0122	0.1008 -0.0123 -3.7285
0.65	-0.1526 0.0878 -0.0055	-0.6510 0.0915 -0.0078	-1.2725 0.0920 -0.0081	-6.2428 0.0924 -0.0083
0.90	0.1000 -0.0296 -0.0093	-0.6671 0.0842 -0.0062	-1.3560 0.0824 -0.0052	-6.8661 0.0810 -0.0037

Table 2: Eigenvalues of the Jacobian matrix (high BGP)

seek to reduce tax evasion too quickly (low ξ), the tax evasion target should be high to prevent low-growth economies to face aggregate instability. However, beyond a certain value of ξ (around 8.2% in our simulations), the government can fix a lower target of tax evasion and remain in a stable equilibrium.

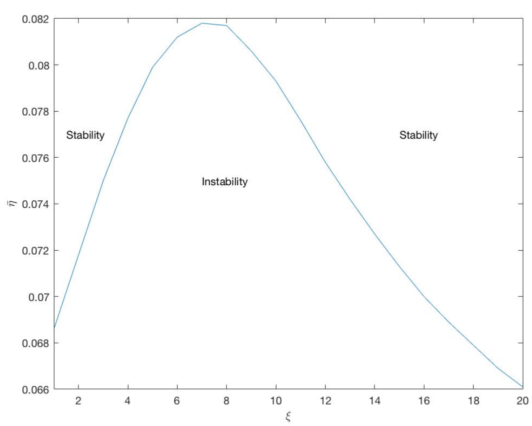

Figure 10: The stability frontier (low BGP)

6. Conclusion

In this paper, we have developed an original theoretical framework to study the relation between tax evasion and public debt accumulation. The economy is described by an endogenous growth model where households strive to evade an exogenous portion of taxes in order to increase their disposable income. The model gives rise to several interesting results. First, the interaction between the intertemporal behavior of households and the government budget constraint generates two BGPs in the long run. This feature has allowed us to analyze the effects of tax evasion on public debt in both low-growth and high-growth economies. We have thus shown that the model exhibits a U-shaped relation between tax evasion and public debt accumulation in high-growth economies while the impact of tax evasion on public debt is always negative in low-growth economies. In high-growth economies, the threshold effect appears because of the two antagonist effects that tax evasion exerts on growth (namely, an increase in the efficiency of the private sector and a decrease in the productivity of public expenditures). By contrast, in low-growth economies, as tax evasion increases, the government must reduce the deficit target to be able to pay the interest on the debt, thereby leading to a decrease in the public debt ratio.

In addition, our model exhibits complex transitional dynamics. Specifically, we have established that the high BGP is always saddle-path stable and that the determinacy of the low BGP depends on the value of the elasticity of the risk premium. Indeed, as the value of the elasticity of the risk premium decreases, the topological behavior of the low BGP moves from local determinacy to local under-determinacy to local over-determinacy. As long as the steady-state is under-determined, the BGP is stable. However, when we cross the point at which the steady-state becomes over-determined, a Hopf bifurcation occurs and aggregate instability appears in the neighborhood of the low BGP. Moreover, we have provided some numerical simulations suggesting that the elasticity of the risk premium should be higher as tax evasion increases in order to prevent low-growth economies to face aggregate instability.

When have then extended the initial setup to consider the optimizing behavior of households towards tax evasion. In this configuration, households make efforts to evade taxes and the government invests resources to combat tax evasion. Moreover, the government follows a rule to reach a certain target of tax evasion. Thus, we have obtained an endogenous expression of tax evasion that depends on the parameters of the model. Generally speaking, we have obtained results that are similar, in the steady-state, to those of the case where tax evasion is exogenous. We find an inverted U-shaped relation between the level of tax evasion targeted by the government and public debt in high-growth economies and a negative impact of the tax evasion target on public debt accumulation in low-growth economies. Regarding transitional dynamics, the high BGP is still locally determined and the stability of low BGP depends on the targeted level of tax evasion and the speed of adjustment between the current and the targeted levels of tax evasion. To illustrate this point, we have provided numerical simulations highlighting the existence of specific combinations of the tax evasion target and the speed of adjustment between the actual and the targeted levels of tax evasion that might generate aggregate instability in low-growth economies.

Further work should clearly deepen the reflection about the optimal structure of public finance for economies with widespread tax evasion. In the spirit of Roubini and Sala-i-Martin (1995), a potential extension would be to introduce seigniorage as an instrument to finance public deficits and productive public expenditures. With such an instrument, governments might choose to increase the seigniorage rate instead of decreasing the deficit target in order to overcome the detrimental consequences of tax evasion, especially in low-growth economies. The introduction of money in the model could also alter the dynamical behavior of the steady-state, which may potentially lead to even more complex dynamics of the balanced-growth paths.

References

Aidt, T., Dutta, J., Sena, V., 2008. Governance regimes, corruption and growth: Theory and evidence. Journal of Comparative Economics 36 (2), 195–220.

Allingham, M. G., Sandmo, A., 1972. Income tax evasion: A theoretical analysis. Journal of public economics 1 (3-4), 323–338.

Barro, R. J., 1990. Government spending in a simple model of endogeneous growth. Journal of Political Economy 98 (5, Part 2), S103–S125.

Cerqueti, R., Coppier, R., 2011. Economic growth, corruption and tax evasion. Economic Modelling 28 (1-2), 489–500.

Chen, B.-L., 2003. Tax evasion in a model of endogenous growth. Review of Economic Dynamics 6 (2), 381–403.

Dimakou, O., 2013. Monetary and fiscal institutional designs. Journal of Comparative Economics 41 (4), 1141–1166.

Futagami, K., Morita, Y., Shibata, A., 1993. Dynamic analysis of an endogenous growth model with public capital. The Scandinavian Journal of Economics, 607–625.

Ghura, D., 1998. Tax revenue in sub-saharan africa: Effects of economic policies and corruption.

Gomes, F., Michaelides, A., Polkovnichenko, V., 2012. Fiscal policy and asset prices with incomplete markets. The Review of Financial Studies 26 (2), 531–566.

Gomme, P., Ravikumar, B., Rupert, P., 2011. The return to capital and the business cycle. Review of Economic Dynamics 14 (2), 262–278.

Huang, H., Wei, S.-J., 2006. Monetary policies for developing countries: The role of institutional quality. Journal of International Economics 70 (1), 239–252.

Imam, P. A., Jacobs, D., 2014. Effect of corruption on tax revenues in the middle east. Review of Middle East Economics and Finance Rev. Middle East Econ. Fin. 10 (1), 1–24.

Litina, A., Palivos, T., et al., 2013. Corruption and tax evasion: Reflections on greek tragedy. Tech. rep., Working Paper, University of Luxeburg and Bank of Greece.

Menuet, M., Minea, A., Villieu, P., 2018. Deficit, monetization, and economic growth: a case for multiplicity and indeterminacy. Economic Theory 65 (4), 819–853.

Minea, A., Villieu, P., 2010. Financial development, institutional quality and maximizing-growth trade-off in government finance. Economic Modelling 27 (1), 324–335.

Minea, A., Villieu, P., 2012. Persistent deficit, growth, and indeterminacy. Macroeconomic Dynamics 16 (S2), 267–283.

Pappa, E., Sajedi, R., Vella, E., 2015. Fiscal consolidation with tax evasion and corruption. Journal of International Economics 96, S56–S75.

Rogoff, K. S., 2017. The Curse of Cash: How Large-Denomination Bills Aid Crime and Tax Evasion and Constrain Monetary Policy. Princeton University Press.

Roubini, N., Sala-i Martin, X., 1995. A growth model of inflation, tax evasion, and financial repression. Journal of Monetary Economics 35 (2), 275–301.

Tanzi, V., Davoodi, H., 1997. Corruption, public investment and growth. In: IMF Working Paper WB/97/139. Citeseer.

Trabandt, M., Uhlig, H., 2011. The laffer curve revisited. Journal of Monetary Economics 58 (4), 305–327.

Appendix A: Resolution of the model

The household's program is solved by maximizing (1) subject to the constraints (2), (3) and (4). The initial values k_0 and b_0 are given and the transversality conditions are standard

$$\lim_{t\to+\infty} \exp(-\rho t)\lambda_{1t} b_t = 0, \tag{A.1}$$

$$\lim_{t\to+\infty} \exp(-\rho t)\lambda_{2t} k_t = 0. \tag{A.2}$$

Thus, the current hamiltonian associated with the household's maximization program is

$$\mathcal{H}_c = u(c_t) + \lambda_{1t}\left\{\left[1-\mathcal{P}(.)\right]R_t b_t + [1-(1-\eta)\tau]y_t - c_t - z_t\right\} + \lambda_{2t} z_t, \tag{A.3}$$

where z_t is a slack variable defined as $z_t = \dot{k}_t$ and λ_{1t} and λ_{2t} are the co-state variables respectively associated with b_t and k_t.

The first order conditions of the household's maximization program with respect to c_t, z_t, b_t and k_t are

$$/c_t \qquad \lambda_{1t} = u'(c_t) = c_t^{-\sigma}, \tag{A.4}$$

$$/z_t \qquad \lambda_{1t} = \lambda_{2t}, \tag{A.5}$$

$$/b_t \qquad \frac{\dot{\lambda}_{1t}}{\lambda_{1t}} = \rho - [1-\mathcal{P}(.)]R_t, \tag{A.6}$$

$$/k_t \qquad \frac{\dot{\lambda}_{2t}}{\lambda_{2t}} = \rho - (1-\alpha)[1-(1-\eta)\tau]\frac{\lambda_{1t}}{\lambda_{2t}} A g_k^{\alpha}, \tag{A.7}$$

The first order conditions have a standard interpretation. Equation (A.5) shows that the shadow price of the financial wealth (λ_{1t}) is equivalent to the shadow price of capital (λ_{2t}) and equation (A.5) highlights that the marginal utility of consumption is equal to the opportunity cost of the financial wealth. In addition, equations (A.6) and (A.7) provide the dynamics of the shadow prices of the financial wealth and the capital stock, respectively. Thus, by differentiating (A.5) and equalizing (A.5) and (A.6), we obtain the tradeoff between government bonds accumulation and private accumulation, as defined in (14).

In the endogenous tax evasion configuration, the representative household also optimizes the efforts to evade taxes. In this case, the current hamiltonian is written as $\mathcal{H}_c = u(c_t) + \lambda_{1t}\left\{\left[1 - \mathcal{P}(.)\right] R_t b_t + [1 - (1 - \eta(e_t))\tau] y_t - c_t - z_t - h_t e_t\right\} + \lambda_{2t} z_t$ and we get a fifth first order condition with respect to e_t

$$h_t = \eta'(e_t)\tau y_t. \tag{A.8}$$

Appendix B: Proof of proposition 4.2

Let us replace y_k^S by its expression in (24) and rewrite the steady-state public debt-to-GDP ratio in the neighborhood of the Solow BGP as follows

$$b_y^S = \left\{ \frac{f_1^S(\eta)\left[f_2^S(\eta) - A^{-1} f_3^S(\eta)\right]}{\rho} \right\}^{\frac{1}{1+\varepsilon}} =: \mathcal{B}(\eta), \tag{B.1}$$

where

$$f_1^S(\eta) = \left[(1-\eta)\tau\right]^{\varepsilon}, \tag{B.2}$$

$$f_2^S(\eta) = (1-\eta)\tau, \tag{B.3}$$

$$f_3^S(\eta) = \left(g_k^S\right)^{1-\alpha} = \left[\frac{\rho}{A(1-\alpha)(1-(1-\eta)\tau)}\right]^{\frac{1-\alpha}{\alpha}}. \tag{B.4}$$

Then, we can determine the first derivatives of equations (B.2)-(B.3) with respect to η

$$f_1'^S(\eta) = -\varepsilon\tau \left[(1-\eta)\tau\right]^{\varepsilon-1} < 0, \tag{B.5}$$

$$f_2'^S(\eta) = -\tau < 0, \tag{B.6}$$

$$f_3'^S(\eta) = -\frac{A(1-\alpha)^2\tau g_k^S}{\alpha\rho} < 0. \tag{B.7}$$

Hence, we can show that the impact of tax evasion on the ratio of public debt-to-GDP is negative in the neighborhood of the Solow BGP

$$\mathcal{B}'(\eta) = \frac{\left(b_y^S\right)^{-1}}{1+\varepsilon} \frac{f_1'^S(\eta)\left[f_2^S(\eta) - A^{-1}f_3^S(\eta)\right] + f_1^S(\eta)\left[f_2'^S(\eta) - A^{-1}f_3'^S(\eta)\right]}{\rho} < 0, \tag{B.8}$$

since $f_2'^S(\eta) - A^{-1}f_3'^S(\eta) < 0$ for positive values of the tax rate.

Appendix C: Proof of proposition 4.3

To study the impact of tax evasion on growth in the neighborhood of the high BGP, we determine the first order condition of (20) with respect to η and extract a critical level of tax evasion (noted $\widehat{\eta}$) such that its long-run effect on growth is reversed

$$\frac{\partial\gamma^B}{\partial\eta} \geq 0 \;\text{ if } \eta \leq \widehat{\eta}, \tag{C.1}$$

where

$$\widehat{\eta} = 1 - \frac{\alpha}{\tau}. \tag{C.2}$$

Following Chen (2003), we assume the tax rate fixed by the government (τ) to be higher than the elasticity of productive public expenditures (α) because of the presence of tax evasion in the economy. Therefore, $0 < \widehat{\eta} < 1$.

In addition, the second order condition shows that the function γ^B is always strictly concave in η. This ensures that the threshold $\widehat{\eta}$ is a maximum

$$\frac{\partial^2 \gamma^B}{\partial \eta^2} = -\frac{\alpha [A(1-\eta)\tau]^{\frac{\alpha}{1-\alpha}} [1 + (1-\eta)\tau - 2\alpha]}{\sigma(1-\alpha)(\eta-1)^2} < 0. \tag{C.3}$$

Appendix D: The reduced form of the model

To obtain a reduced form of the model, let us define the growing variables in intensive terms: $c_k = c_t/k_t$, $g_k = g_t/k_t$, $b_k = b_t/k_t$ and $y_k = y_t/k_t$. From the Keynes-Ramsey rule, we determine

$$\frac{\dot{c}_k}{c_k} = \frac{r_t - \rho}{\sigma} - \gamma_k, \tag{D.1}$$

where γ_k, that corresponds to the growth rate of private capital ($\gamma_k := \dot{k}_t/k_t$), is obtained using the IS equilibrium

$$\gamma_k = y_k - g_k - c_k. \tag{D.2}$$

The local stability of the model can be studied from a reduced-form in c_k and g_k or in c_k and b_k. We successively present both alternatives. For simplicity, we will resort to the reduced-form in c_k and g_k to study the local stability of the low BGP and to the reduced-form in c_k and b_k for the local stability of the high BGP.

Appendix D.1: The reduced form in c_k and g_k

The government budget constraint provides a first relation describing the evolution of the ratio of public debt-to-capital

$$\frac{\dot{b}_k}{b_k} = \frac{1}{1+\varepsilon}\left[\varepsilon\alpha - \frac{(1-\alpha)g_k}{(\theta + (1-\eta)\tau)y_k - g_k}\right]\frac{\dot{g}_k}{g_k}. \tag{D.3}$$

In addition, from (8), we obtain a second expression of the dynamics of b_k over time

$$\frac{\dot{b}_k}{b_k} = \frac{\theta y_k}{b_k} - \gamma_k. \tag{D.4}$$

Combing (D.3) and (D.4), we can easily extract the dynamics of the ratio of productive public expenditures-to-capital

$$\frac{\dot{g}_k}{g_k} = \frac{(1+\varepsilon)\left[(\theta+(1-\eta)\tau)y_k - g_k\right]\left[\frac{\theta y_k}{b_k} - \gamma_k\right]}{\varepsilon\alpha\left[(\theta+(1-\eta)\tau)y_k - g_k\right] - (1-\alpha)g_k}. \tag{D.5}$$

Hence, replacing γ_k by its expression, we obtain the following reduced-form in c_k and g_k

$$\begin{cases} \dot{c}_k = \left(\dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k\right) c_k, \\ \dot{g}_k = \mathcal{M}(g_k)\left[\dfrac{\theta A g_k^{\alpha}}{b_k} + c_k + g_k - y_k\right], \end{cases} \tag{D.6}$$

where

$$\mathcal{M}(g_k) = \frac{(1+\varepsilon)[(\theta+(1-\eta)\tau)y_k - g_k]g_k}{\varepsilon\alpha[(\theta+(1-\eta)\tau)y_k - g_k] - (1-\alpha)g_k}. \tag{D.7}$$

Appendix D.2: The reduced form in c_k and b_k

We obtain the reduced-form in c_k and b_k from equations (8) and (9). After some manipulations, we get

$$\begin{cases} \dot{c}_k = \left(\dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k\right) c_k, \\ \dot{b}_k = \theta y_k + (c_k + g_k - y_k)\, b_k, \end{cases} \tag{D.8}$$

where g_k is an implicit function of b_k ($g_k := \Psi(b_k)$).

Appendix E: Local stability of the BGPs

In the neighborhood the low BGP (for $\theta \to 0$), the Jacobian matrix associated with the system (D.6) is

$$\mathbf{J^S} = \begin{bmatrix} c_k^S & \left[1 - \alpha A\left(g_k^S\right)^{\alpha-1}(1-(1-\alpha)(1-(1-\eta)\tau)\sigma^{-1})\right] c_k^S, \\ \tilde{\mathcal{M}}\left(g_k^S\right) & \tilde{\mathcal{M}}\left(g_k^S\right)\left[1 - \alpha A\left(g_k^S\right)^{\alpha-1}\right] \end{bmatrix}. \tag{E.1}$$

From (E.1), we easily obtain $\mathcal{D}(\mathbf{J}^S) = -\tilde{\mathcal{M}}\left(g_k^S\right) c_k^S (1-\alpha)\alpha(1-(1-\eta)\tau)\sigma^{-1} A \left(g_k^S\right)^{\alpha-1}$ and $\mathcal{T}(\mathbf{J}^S) = c_k^S + \tilde{\mathcal{M}}\left(g_k^S\right)\left[1 - \alpha A \left(g_k^S\right)^{\alpha-1}\right]$ where $\tilde{\mathcal{M}}(g_k^S)$ corresponds to $\mathcal{M}(g_k)$ in the neighborhood of the low steady-state for $\theta \to 0$

$$\tilde{\mathcal{M}}(g_k^S) = \frac{(1+\varepsilon)[(1-\eta)\tau y_k^S - g_k^S]g_k^S}{\varepsilon\alpha[(1-\eta)\tau y_k^S - g_k^S] - (1-\alpha)g_k^S}. \tag{E.2}$$

Notice that since the growth rate in the neighborhood of the low BGP tends towards 0 ($\gamma^S = 0$), the consumption-to-capital ratio is expressed as : $c_k^S = y_k^S - g_k^S$.

In the neighborhood the high BGP (for $\theta \to 0$), the Jacobian matrix associated with the system (D.8) is

$$\mathbf{J^B} = \begin{bmatrix} c_k^B & 0 \\ 0 & -\gamma^B \end{bmatrix}. \tag{E.3}$$

Hence, it is clear that the determinant of (E.3) is always strictly negative ($\mathcal{D}(\mathbf{J^B}) = -\gamma^B c_k^B < 0$).

Appendix F: The steady-state solution of the model with endogenous tax evasion

In the endogenous tax evasion case, the steady-state solution of the model can be obtained at the intersection of the following two relations between long run growth and the public debt-to-capital ratio. The first relation is derived from the definition of the public debt-to-GDP target while the second relation comes from the government budget constraint

$$b_k^1(\gamma) = \frac{\theta y_k}{\gamma}, \tag{F.1}$$

$$b_k^2(\gamma) = y_k \left\{ \frac{[\theta + (1-(1-\beta)\eta(e))\tau]y_k - g_k}{[(1-\alpha)(1-(1-\eta(e))\tau)]\, y_k^2\, [(1-\eta(e))\tau]^{-\varepsilon}} \right\}^{\frac{1}{1+\varepsilon}}, \tag{F.2}$$

where

$$g_k = \left\{ \frac{\gamma\sigma + \rho}{A(1-\alpha)[1-(1-\eta(e))\tau]} \right\}^{\frac{1}{\alpha}}, \tag{F.3}$$

and

$$\eta(e) = \frac{\gamma}{\xi} + \bar{\eta}. \tag{F.4}$$

The endogenization of tax evasion makes the derivation of analytical results in the steady-state rather difficult. Therefore, we implement numerical simulations and show, in figure 4, that the behavior of the economy in the steady-state is similar to the one in which tax evasion is exogenous.

Appendix G: The model with endogenous tax evasion: reduced form and topological behavior of the BGPs

Outside the steady-state, the model with endogenous tax evasion gives rise to a three-variable reduced form in c_k, h_k and g_k

$$\begin{cases} \dot{c}_k = \left(\dfrac{r_t - \rho}{\sigma} + c_k + g_k - y_k \right) c_k, \\ \dot{h}_k = \left[\xi \left(\eta(e_t) - \bar{\eta} \right) + c_k + g_k - y_k \right] h_k, \\ \dot{g}_k = \mathcal{K}(.) \left\{ (1+\varepsilon) \left[\dfrac{\theta y_k}{b_k} + c_k + g_k - y_k \right] - x(.) \dfrac{\dot{h}_k}{h_k} \right\}, \end{cases} \tag{G.1}$$

where $r_t = (1-\alpha)[1-(1-\eta(e_t))\tau]Ag_k^{\alpha}$ with $\eta(e_t) = \eta_0 e_t^{\beta}$ and the expression of e_t is given by

$$e_t = \left(\frac{\eta_0 \beta \tau y_k}{h_k} \right)^{\frac{1}{1-\beta}}. \tag{G.2}$$

In addition

$$\mathcal{K}(.) = \frac{[(\theta + (1-\eta(e_t))\tau)y_k - g_k + e_t h_k]g_k}{[(\theta + (1-\eta(e_t))\tau)y_k - g_k + e_t h_k]\left[\varepsilon\alpha - \alpha\left(\Psi^1(.) + \Psi^2(.)\right)\right] - (1-\alpha)g_k + e_t h_k}, \tag{G.3}$$

$$x(.) = \Psi^1(.) + \Psi^2(.) + \frac{e_t h_k}{(\theta + (1-\eta(e_t))\tau)y_k - g_k + e_t h_k}, \tag{G.4}$$

where

$$\Psi^1(.) = \frac{\beta\tau\eta(e_t)}{(1-\beta)(1-(1-\eta(e_t))\tau)} \quad \text{and} \quad \Psi^2(.) = \frac{\beta\varepsilon\eta(e_t)}{(1-\beta)(1-\eta(e_t))}$$

.

Finally, the government budget constraint provides the new expression of the public-debt to capital ratio

$$b_k = y_k \left\{ \frac{[\theta + (1-(1-\beta)\eta(e_t))\tau]y_k - g_k}{[(1-\alpha)(1-(1-\eta(e_t))\tau)]\, y_k^2 \, [(1-\eta(e_t))\tau]^{-\varepsilon}} \right\}^{\frac{1}{1+\varepsilon}}. \tag{G.5}$$

In order to examine the local stability of the BGPs, we linearize the system G.1 in the neighborhood of the steady-state i (where $i \in (L, H)$)

$$\begin{pmatrix} \dot{c}_k \\ \dot{h}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}^i \begin{pmatrix} c_k - c_k^i \\ h_k - h_k^i \\ g_k - g_k^i \end{pmatrix}, \tag{G.6}$$

Let us first define the derivatives of $\eta(e_t)$, r_t, $\Psi^1(.)$, $\Psi^2(.)$ and $x(.)$ with respect to g_k and h_k

$$\eta_g^i := \left.\frac{\partial\eta(e_t)}{\partial g_k}\right|_{*i} = \frac{\alpha\beta}{(1-\beta)}\frac{\eta(e^i)}{g_k^i}, \quad \eta_h^i := \left.\frac{\partial\eta(e_t)}{\partial h_k}\right|_{*i} = -\frac{\beta}{(1-\beta)}\frac{\eta(e^i)}{h_k^i}, r_h^i := \left.\frac{\partial r_t}{\partial h_k}\right|_{*i} = -\left(\frac{1-\alpha}{1-\beta}\right)e^i,$$

$$r_g^i := \left.\frac{\partial r_t}{\partial g_k}\right|_{*i} = \alpha(1-\alpha)\left[\left(\frac{1-\alpha}{1-\beta}\right)\frac{h_k^i}{g_k^i}e^i + \left(1-\left(1-\eta\left(e^i\right)\right)\tau\right)A\left(g_k^i\right)^{\alpha-1}\right],$$

$$\tilde{\Psi}_g^1(.) := \left.\frac{\partial\Psi^1(.)}{\partial g_k}\right|_{*i} = \frac{\beta\tau}{1-\beta}\frac{(1-\tau)\eta_g^i}{(1-(1-\eta(e^i))\tau)^2}, \quad \tilde{\Psi}_g^2(.) := \left.\frac{\partial\Psi^2(.)}{\partial g_k}\right|_{*i} = \frac{\beta\varepsilon\eta_g^i}{(1-\beta)(1-\eta(e^i))^2},$$

$$\tilde{\Psi}_h^1(.) := \left.\frac{\partial\Psi^1(.)}{\partial h_k}\right|_{*i} = \frac{\beta\tau}{1-\beta}\frac{(1-\tau)\eta_h^i}{(1-(1-\eta(e^i))\tau)^2}, \quad \tilde{\Psi}_h^2(.) := \left.\frac{\partial\Psi^2(.)}{\partial h_k}\right|_{*i} = \frac{\beta\varepsilon\eta_h^i}{(1-\beta)(1-\eta(e^i))^2},$$

$$\tilde{x}_g(.) := \left.\frac{\partial x(.)}{\partial g_k}\right|_{*i} = \tilde{\Psi}_g^1(.) + \tilde{\Psi}_g^2(.) + \frac{e^i}{1-\beta}\frac{h_k^i}{g_k^i}\frac{\alpha\left[1-(1-\beta)\eta\left(e^i\right)\right]\tau y_k^i + (1-\alpha-\beta)g_k^i}{(1-\eta(e^i))\tau y_k^i - g_k + e^i h_k^i},$$

$$\tilde{x}_h(.) := \left.\frac{\partial x(.)}{\partial h_k}\right|_{*i} = \tilde{\Psi}^1_h(.) + \tilde{\Psi}^2_h(.) + \frac{\beta\left(g^i_k - \tau y^i_k\right)e^i}{(1-\beta)\left[\left(1-\eta\left(e^i\right)\right)\tau y^i_k - g^i_k + e^i h^i_k\right]}.$$

The Jacobian matrix of the system G.1 in the neighborhood of the steady-state i can be written as

$$\mathbf{J^i} = \begin{bmatrix} c^i_k & \sigma^{-1} r^i_h c^i_k & C^i_g \\ h^i_k & \xi \eta^i_h h^i_k & H^i_g \\ G^i_C & G^i_h & G^i_g \end{bmatrix} \tag{G.7}$$

where

$$C^i_g := \left.\frac{\partial \dot{c}_k}{\partial g_k}\right|_{*i} = \left[1 + \sigma^{-1} r^i_g - \alpha A \left(g^i_k\right)^{\alpha-1}\right] c^i_k$$

$$H^i_g := \left.\frac{\partial \dot{h}_k}{\partial g_k}\right|_{*i} = \left[1 + \xi \eta^i_g - \alpha A \left(g^i_k\right)^{\alpha-1}\right] h^i_k$$

$$G^i_c := \left.\frac{\partial \dot{g}_k}{\partial c_k}\right|_{*i} = \mathcal{K}^i(.)\left[(1+\varepsilon) - x^i(.)\right]$$

$$G^i_h := \left.\frac{\partial \dot{g}_k}{\partial h_k}\right|_{*i} = -\mathcal{K}^i(.)\left[\tilde{x}^i_h(.)\gamma + x^i(.)\xi\eta^i_h\right]$$

$$G^i_g := \left.\frac{\partial \dot{g}_k}{\partial g_k}\right|_{*i} = \mathcal{K}^i(.)\left[(1+\varepsilon)\left(1 - \alpha A\left(g^i_k\right)^{\alpha-1}\right) - \tilde{x}^i_g(.)\gamma^i - x(.)\left(1 + \xi\eta^i_g - \alpha A\left(g^i_k\right)^{\alpha-1}\right)\right]$$

Interestingly, we can observe that when $\eta_0 \to 0$, $e_t = \eta(e_t) = \Psi^1(.) = \Psi^2(.) = x(.) = 0$ and $\mathcal{K}(.) \to (1+\varepsilon)^{-1}\,\mathcal{M}(g^i_k)|_{\eta\to 0}$.