

Production Externalities and Investment Caps: a Welfare Analysis under Uncertainty

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Abstract

In markets where production has adverse externalities, policy makers may wish to increase welfare by imposing a cap on market entries. In this paper, we examine the implications that the cap has on the firms' investment equilibrium policy and on social welfare in the presence of market uncertainty. In contrast with previous literature, we explicitly model the present externality and then let the social planner choose the cap level maximizing welfare. We find that: i) if the consideration of the option value triggers investment at price above the social marginal cost of production, then it is optimal to have no cap at all; ii) otherwise, the cap should be set on the current market quantity and a ban on further market entries should be announced.

Key words: Investment, Uncertainty, Caps, Competition, Externalities, Welfare.

JEL codes: C61, D41, D62

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1. Introduction

Caps exist in a variety of markets and activities – from fishing¹ to trade and foreign investment, taxis and all the way to immigration. The discussion concerning their economic impact has recently been reignited by President Trump’s statements concerning the possibility of US withdrawing from international trade agreements, which, in the past two decades, have eliminated many import quotas (The Economist, 2018), and the need of tightening immigration quotas² (The Economist, 2017).

One of the main reasons why caps are used is that, as economic intuition suggests, if production costs are not fully internalized by producers, a cap on the aggregated market quantity may increase social welfare. This is because the cap prevents production from reaching the range where output is sold at a price not covering its social cost of production. This may be the case when, for instance, a negative externality such as pollution is associated with production, or when the entry of foreign firms negatively impacts the performance of the domestic industry³.

The seminal work by Bartolini (1993) was the first study of how a cap impacts the market equilibrium within a dynamic setting characterized by uncertain market conditions and irreversible investment prior to production. His analysis has shown that the existence of the cap leads to investment dynamics which are profoundly

¹ See e.g. Birkenbach et al. (2017).

² Note that the term “quota” usually stands for the share of a total assigned to a specific group (see e.g. OECD, 2006). Hence, in terms of actual impact, setting a quota is equivalent to setting a cap on the migration level allowed to the targeted group.

³ See e.g. Chao and Eden (2003) on the adoption of foreign investment quotas, Calzolari and Lambertini (2007) on the use of voluntary export restraints and Pennings (2005) on the impact of tariffs, corporate taxes and investment subsidies.

different from those usually portrayed by the literature on irreversible investment under uncertainty. This literature, summarized by Dixit and Pindyck (1994), has shown that in a decentralized setting, when no cap is present, firms invest (in order to enter the market) sequentially. It has also shown that due to the uncertain future profits and the irreversibility of their initial investment, firms invest only when the output price is sufficiently above the marginal cost of production in order to account for the option value which is implicitly lost once investment takes place. Differently, as Bartolini (1993) firstly showed, introducing a cap on the aggregate market quantity gives rise at a certain point in time to a “competitive run”. Firms enter the market sequentially only up to a certain point in time, and then a run starts, exhausting at once the residual investment slots. During the run, output is sold at a price below its marginal cost of production. This is because firms fear that they will lose their entry option while postponing investment in order to wait for higher prices. Note in fact that, once the cap becomes binding, their option to invest is worthless. Thus, they abstract from the consideration of the option value associated with the investment decision and rush to enter the market. The run reduces welfare as it brings on the market additional output at once and at a price below the marginal cost of production

Several studies (which we briefly review later in this section) have followed Bartolini (1993) in analyzing the market equilibrium with a cap on aggregate quantity under investment irreversibility and market uncertainty. Both Bartolini (1993) and these follow-up studies motivate the introduction of the cap on the basis of certain welfare gains that could be achieved by restricting private economic actions and agree on the welfare loss induced by the run. They all take the level of the cap as exogenously set or, in Bartolini’s words, as the “*result of a more general political equilibrium*”

(Bartolini, 1995), where the harmful impact of private economic actions is taken into account. This makes, as one can immediately see, the welfare analysis not conclusive and poses the challenge of completing it by, first, being explicit in the welfare function about the externalities due to private economic actions. Hence, endogenizing the cap by setting it at a level maximizing the welfare function is a second natural step.⁴ As far as we know, no research has addressed this issue, a challenge that motivates and sets the focus of our paper.

In this paper, we set up a model analyzing market entry in the presence of a cap on aggregate investment where we explicitly consider the presence of an externality associated with private investment. We then determine the optimal investment policy set by private firms acting in a decentralized setting and the optimal level of the cap to be set by a planner maximizing the welfare associated to the considered market.

Our main findings are as follows. First, we identify the circumstances under which the output price triggering firms' entry is above the social marginal cost of production, due to the consideration of the option value associated with the investment decision and in spite of the externality. The intuition is straightforward. Assume, for example, that only the 80% of the social marginal cost falls on the producers, while, once taken into account the option value, a firm should enter the market only when the output price is 50% higher than the private marginal cost. Hence, market entry occurs only when the output price is 20% higher than the social marginal cost. Thus, as this additional investment leads to a welfare gain, a welfare-maximizing planner should

⁴ Note that this issue is explicitly raised by Bartolini (1993, Section 2.1).

not introduce any cap on market entries. In contrast, when the “uncertainty premium” does not allow counterbalancing the externality, entry would occur at a price that, even if above the private marginal cost, is below the social one. When this is the case, as welfare is decreasing in the level of aggregate investment, stopping further entries is the optimal policy. Note that in this case the optimal policy depends on the history of the targeted market. In fact, as some market capacity may already exist when the introduction of a cap is considered and this capacity cannot be removed, the optimal cap to be set is bounded from below and must be set equal to the lowest feasible level, that is, the market capacity already present. As one can immediately see, our first finding has relevant implications for the design of policies contemplating the introduction of a cap. We show in fact that the introduction of a cap is not consistent with its assumed target of maximizing welfare, as, depending on the circumstances, it deters welfare from increasing or, if the planner sets a cap allowing for further market entries, it decreases welfare.

Second, we show that, apart from the welfare implications above, the introduction of a cap does not have, from the perspective of the social planner, any positive implication when considering the timing of market entries. In fact, first, firms, when entering sequentially, keep following the investment policy that they would set in the absence of the cap, a policy which results from the mere consideration of the production cost privately perceived. Second, the introduction of the cap induces a competitive run that, irrespective of the severity of the externality, produces welfare losses. This is because first, during the run, investment occurs when the output price is, due to the fear of being excluded from the market, always below the social marginal cost of

production, and second, because these welfare losses weight more in present terms as they materialize at an earlier date.

In a later study, Bartolini (1995) has shown that the run and its resulting welfare loss can be avoided by rationing the right to enter the market. Specifically, he shows that if licenses are distributed among firms when the cap is announced, the fear of losing their entry option vanishes and firms enter the market sequentially until the cap becomes binding. Yet, the model in that article does not contain an explicit modeling of the externality which motivates imposing the cap, and therefore cannot search for the optimal size of the cap. Instead it takes the size of the cap as exogenous.

We analyze both the case where licenses are issued and the run does not emerge, and the opposite case where entry prior to hitting the cap is free. In both cases we find the same optimal policy specified earlier regarding the size of the cap, that is, to have no cap at all if the externality is not severe enough or, otherwise, to stop immediately further entries. This happens because the two cases share two of the main welfare losses from a non-optimal cap: (i) allowing the production of welfare-harming units in the case where the externality is sufficiently severe; (ii) preventing the production of welfare-enhancing units in the case where the externality is not severe enough. Not having the run in the rationing case is relevant therefore only to the level of welfare under a non-optimal cap (which is higher under licensing on account of not having the run) but not to the optimal policy.

The following table summarizes our results:

Case Issue	The externality dominates the uncertainty premium	The uncertainty premium dominates the externality
Welfare losses/gains with a non-optimal cap (before the run)	Additional output sold at a price above the private marginal cost but below the social marginal cost of production.	Additional output sold at a price above the social marginal cost of production.
Welfare losses with a non-optimal cap (during the run)	Additional output sold at a price below the private marginal cost of production.	Additional output sold at a price below the private marginal cost of production.
Optimal policy	Stopping further market entries	No cap on aggregate investment

Table 1: Summary of results

Among the other studies that have adopted the Bartolini (1993) framework, the most prominent ones are, Moretto and Vergalli (2010) and Di Corato, Moretto and Vergalli (2013).

Moretto and Vergalli (2010) study how a cap on immigration set by the government of a host country affects the decision of potential migrants. They view migration as an irreversible investment with uncertain future benefits and study its timing in a real-options framework. Migration occurs when its profitability, mostly based on the labor market conditions in the host country, is high enough to compensate for the migration cost and for the value of the option to migrate, which is implicitly lost once left the home country. If the host country wishes to control immigration by introducing a cap, the Bartolini's dynamic pattern emerges with a run starting at a certain point in time. Last, the authors show that the government may delay the "cap-attack" by creating uncertainty about the size of the cap.

Di Corato, Moretto and Vergalli (2013) study the dynamic characterizing the conversion of forestland into agricultural land in the short and in the long run. Forest

conservation secures the provision of public goods and services for which private landowners are not compensated. Their conversion decisions do not take into account the induced welfare loss. This motivates the introduction of cap on the aggregate amount of developable land. Also here the Bartolini's dynamic pattern emerges and the run leads to the complete exhaustion of the targeted extent of developable land at a socially suboptimal pace. The study then considers feasible combinations of second-best policy tools which may reduce the speed of the conversion process.

The paper remainder is as follows. In Section 2, we present the basic model, i.e. the no policy scenario, and identify the optimal investment policy in equilibrium and the associated level of welfare. Section 3 considers the scenario where a cap on aggregate investment is introduced in a market where firms may freely enter. We present the optimal investment policy, characterize and discuss the emergence of a competitive run and determine the welfare-maximizing cap. Section 4 considers the scenario where a cap is introduced but the right to enter is rationed. We determine the optimal investment policy and the welfare-maximizing cap and compare our findings with the ones obtained when studying the case of free entry. In Section 5, we illustrate our result by proposing two numerical exercises. In the first exercise we show the impact of a cap on welfare by comparing welfare accruing under free entry and under rationing with the welfare in a no policy scenario and with the welfare accruing when following the policies suggested in this paper. In the second exercise we focus on the suboptimality of the firms' entry timing from a social perspective. This is done by calculating the expected amount of time by which entries are accelerated with respect to the entry timing that would be set in a first-best scenario. Section 6 concludes. The Appendix contains the proofs omitted from the text.

2. The basic model

Within a continuous time setting, consider a market for a good named A whose demand at each point in time is given by:

$$(1) \quad P_t = \frac{X_t}{Q_t},$$

where P_t and Q_t are the price and quantity of good A at time t , respectively. The state of demand, X_t , changes stochastically over time according to the following Geometric Brownian Motion:

$$(2) \quad dX_t = \mu \cdot X_t \cdot dt + \sigma \cdot X_t \cdot dZ_t,$$

where μ and σ are constants which measure the drift and the variance of X_t , respectively, and dZ_t is the increment of the standard Wiener process satisfying at each t the following conditions:

$$(3) \quad E(dZ_t) = 0, \quad E(dZ_t)^2 = dt$$

All firms have an infinitesimally small productive capacity, ΔQ , allowing them to produce one unit of A. Each firm, once entered the market, commits to offer permanently one unit of A. Thus, Q_t represents both the market quantity of good A and the number of active firms.

All firms face the same cost structure in which, at each point in time, producing one unit of A entails an instantaneous social cost equal to $M > 0$. We assume that part of this cost is an externality that firms do not incur. Therefore, their instantaneous cost of production is equal to $\lambda \cdot M$, where $0 < \lambda < 1$.

Firms are risk-neutral and r denotes the rate at which they discount future payoffs. As standard in the literature, we assume that $r > \mu$ to secure the convergence of the firm's value.⁵ Last, note that, in the economy of our model, the present value of the flow of operating costs, $\frac{\lambda \cdot M}{r}$, may equivalently be viewed as an irreversible investment cost to be paid when entering the market.

2.1 Optimal investment in the absence of a cap

Under the setup described above, our model of optimal investment in the absence of government intervention is a specific case of the one analyzed in Chapter 8 of Dixit and Pindyck (1994). Thus, in this sub-section, we illustrate it following their analysis.

At each instant, each idle firm has to decide whether to enter the market in order to produce and supply an additional unit of good A, or not. This decision depends on the expected profitability of the investment effort. Therefore, given the current⁶ Q , investment takes place only when X is large enough to secure an expected flow of revenues covering the flow of operating costs plus the value of the option to wait, which is implicitly lost once invested.

⁵ See appendix A where the value of the firm is determined.

⁶ In the following, we will drop the time subscript for notational convenience.

Let $V(Q, X)$ be the value of a firm active in the market. By a standard no-arbitrage analysis⁷, this value is equal to:

$$(4) \quad V(Q, X) = Y(Q) \cdot X^\beta + \frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r}$$

where $\beta > 1$ is the positive root of the following quadratic equation:

$$(5) \quad \frac{1}{2} \cdot \sigma^2 \cdot x^2 + \left(\mu - \frac{1}{2} \cdot \sigma^2 \right) \cdot x - r = 0.$$

Note that by properties of the Geometrical Brownian motion, if Q will remain forever at its current level, then:

$$E \left[\int_{t=0}^{\infty} \left(\frac{X}{Q} - \lambda \cdot M \right) \cdot e^{-r \cdot t} \cdot dt \right] = \frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r}$$

which means that the second and third terms of (4) represent the expected present value of the flow of the firm's future profits if Q remains forever at its current level. The first term in the RHS of (4) accounts therefore for how future changes in the supplied quantity Q , due to further market entries, affect the value of the firm.

Now, let denote the investment threshold function by $X^*(Q)$. As standard, some boundary conditions are required for its determination. The first one is the following *Value Matching Condition*:

⁷ See Appendix A.

$$(6) \quad V[Q, X^*(Q)] = 0.$$

The second one is the following *Smooth Pasting Condition*:

$$(7) \quad V_X[Q, X^*(Q)] = 0.$$

Substituting (4) in (6) and (7) yields:

$$(8) \quad X^*(Q) = \bar{\beta} \cdot (r - \mu) \cdot \lambda \cdot \frac{M}{r} \cdot Q,$$

where $\bar{\beta} \equiv \frac{\beta}{\beta-1} > 1$ is the wedge that, as standard in the real options literature, takes the presence of uncertainty and investment irreversibility into account (see e.g. Dixit and Pindyck, 1994, Ch. 5).

Note that $X^*(Q)$ is an increasing function of Q . This makes sense considering that the larger Q , the higher market competition and, ceteris paribus, the higher the expected profitability required for triggering further investment.

2.2 Welfare in the absence of a cap

From a static perspective, the most basic intuition about the welfare that could be associated to the market for good A is that if the output price exceeds the social marginal cost of production, adding another unit to the market raises welfare. We

open this sub-section by bringing this intuition into our model and comparing the output price, P , with the marginal social cost, M , for the case where $\mu = 0$. Note that this is only to cling stronger to the static case. Using (1) and (8), we find that in this case the price level triggering additional investments is $P^* = \frac{X^*(Q)}{Q} = \bar{\beta} \cdot \lambda \cdot M$. Hence, it follows that if $\bar{\beta} \cdot \lambda > 1$, investments take place when $P^* > M$ and welfare increases as the output price is higher than the marginal social cost of production. In contrast, when $\bar{\beta} \cdot \lambda < 1$, welfare decreases as investments take place when the output price is below the marginal social cost of production. No gains in welfare may instead be associated to additional investments when $\bar{\beta} \cdot \lambda = 1$. Thus, while, on the one hand, the presence of an investment externality, illustrated by the parameter $\lambda < 1$, may induce investment which is suboptimal from a social perspective, the wedge $\bar{\beta} > 1$ may trigger it at a price high enough to counterbalance the effect of the externality and secure a welfare gain.

The deep roots of this intuition make it dominant also in the dynamic and stochastic model analyzed here, as the formal derivation of the welfare function conducted in this subsection shows. In particular, we find here that $\bar{\beta} \cdot \lambda > 1$ is indeed the condition for welfare to rise, despite the externality, when an additional unit of good A is supplied.

After presenting this intuition, we now turn to the formal derivation of the welfare associated to the market for good A. This will be done following Bartolini (1995). Via the same procedure illustrated in detail in Appendix A (to determine the value of an

active firm), we find that the expected discounted social welfare, given the current levels of X and Q , is:

$$(9) \quad W(Q, X) = C(Q) \cdot X^\beta + \frac{X \cdot \ln(Q)}{r - \mu} - \frac{M \cdot Q}{r},$$

In (9), we subtract the discounted social cost associated with the production of Q units of good A, i.e. $\frac{M \cdot Q}{r}$, from the surplus resulting from the supply of those units, i.e. $\frac{X \cdot \ln(Q)}{r - \mu}$, while the first term stands for the expected discounted flow of net surplus associated with future market entries. In order to determine the term $C(Q)$, we impose that the following *Value Matching Condition*:

$$(10) \quad W_Q[Q, X^*(Q)] = 0,$$

holds at the investment threshold $X^*(Q)$.

Substituting (9) in (10), partially differentiating with respect to Q , using (8), and rearranging terms, yields:

$$(11) \quad C'(Q) = K \cdot \frac{1 - \bar{\beta} \cdot \lambda}{Q^\beta},$$

where:

$$(12) \quad K \equiv \frac{1}{\left(\frac{M}{r}\right)^{\beta-1} \cdot [\lambda \cdot \bar{\beta} \cdot (r - \mu)]^\beta} > 0.$$

Integrating on both sides of (11) yields:

$$(13) \quad C(Q) = \frac{K}{\beta-1} \cdot \frac{\bar{\beta} \cdot \lambda - 1}{Q^{\beta-1}} + G,$$

To find the value of the term G , note that when $Q \rightarrow \infty$ the threshold $X^*(Q)$ goes to infinity as well. This implies that, as the probability of X hitting the threshold goes to 0, no further changes in Q are expected, and therefore the value associated with such changes is zero. Formally put:

$$(14) \quad \lim_{Q \rightarrow \infty} C(Q) = 0.$$

Since $\beta > 1$, (14) and (13) imply that $G = 0$, and therefore:

$$(15) \quad C(Q) = \frac{K}{\beta-1} \cdot \frac{\bar{\beta} \cdot \lambda - 1}{Q^{\beta-1}}.$$

Thus, as the intuition presented in the beginning of this sub-section hinted, the necessary and sufficient condition for welfare to rise when new firms enter the market is $\bar{\beta} \cdot \lambda > 1$.

3. Optimal investment with a cap and free entry

The possibility that $C(Q)$ may be negative could lead policy makers to consider the limitation of future investments by setting a cap, \bar{Q} , on the level of aggregate

investment Q . In this section we analyze this case following Bartolini (1993). Similarly to the analysis conducted in Section 2.1, we start by considering the value of a firm contemplating market entry, $V(Q, X)$, as defined in (4). Then, in order to find the threshold function $X^*(Q)$, we use the *Value Matching Condition* (6). From here on the analysis departs from that conducted above, as Condition (7) should be replaced by the following boundary condition:

$$(16) \quad v_Q[Q, X^*(Q)] = 0$$

Bartolini (1993) proves the existence of Condition (16) in *Proposition 1* of his article. As he shows there, the condition springs from:

$$(17) \quad v[Q, X^*(Q + \Delta Q)] = v[Q + \Delta Q, X^*(Q + \Delta Q)] .$$

Condition (17) shows that when the quantity is Q and X hits the corresponding threshold level, $X^*(Q)$, then, by the definition of $X^*(Q)$ as a threshold level, Q is increased by an increment ΔQ with probability 1. This consideration, together with the no-arbitrage condition, explains the equality in (17) between the values corresponding to the two states. Dividing on both sides by ΔQ and taking the limit $\Delta Q \rightarrow 0$ leads to (16). Note that (6) and (16) are not optimality conditions. They should hold for any $X^*(Q)$, not necessarily the optimal one, as they merely reflect, given a certain threshold, the no-arbitrage condition set on the value of the firm. This means that (6) holds for any level of Q and that, in turn, once taken the derivative with respect to Q on both its sides,

$$(18) \quad \frac{dV[Q, X^*(Q)]}{dQ} = 0.$$

Expanding (18) and using (16) yield the condition:

$$(19) \quad V_X[Q, X^*(Q)] \cdot \frac{dX^*(Q)}{dQ} = 0$$

For (19) to hold, one should have either $V_X[Q, X^*(Q)] = 0$ or $\frac{dX^*(Q)}{dQ} = 0$. In the former case, as one may easily verify, the *Smooth Pasting Condition* (7) holds, and the investment threshold function $X^*(Q)$ is, as in the case where no cap is present, given by (8).

In the latter case, the *Smooth Pasting Condition* (7) does not hold and, consequently, the threshold function (8) does not apply. To understand how this case should be treated, we recall that $Y(Q)$ represents how future changes in Q affect the value of the firm. when Q is already at the cap, no such changes can occur and consequently the value associated to these variations is 0. Thus, formally put, we have:

$$(20) \quad Y(\bar{Q}) = 0.$$

Using (20), (4) and (6), we have that when Q reaches the cap, \bar{Q} , the investment threshold is:

$$(21) \quad \bar{X} = (r - \mu) \cdot \lambda \cdot \frac{M}{r} \cdot \bar{Q}.$$

Since the *Smooth Pasting Condition* (7) does not hold in \bar{Q} then, by continuity, it also does not hold within the interval $[\tilde{Q}, \bar{Q})$ where, as $\frac{dX^*(Q)}{dQ} = 0$, \tilde{Q} satisfies:

$$(22) \quad X^*(\tilde{Q}) = \bar{X}.$$

Substituting (8) and (21) into (22) yields:

$$(23) \quad \tilde{Q} = \frac{1}{\beta} \cdot \bar{Q}.$$

Summing up, the optimal investment policy is:

- as long as $Q < \tilde{Q}$, investment occurs whenever X hits the threshold $X^*(Q)$.

The rising Q makes $X^*(Q)$ rise too, so that X is once again below the threshold and further investment is postponed until the first time X hits the threshold. In *Figure 1* below this is described by the move from point E to point F;

- if $Q = \tilde{Q}$, then when X hits the threshold $X^*(Q)$ investment occurs but, in contrast with what would happen in the interval $Q < \tilde{Q}$, the threshold is not increased by the rising Q as $\frac{dX^*(Q)}{dQ} = 0$. Thus, X is still at the threshold and

investment continues until the cap becomes binding. In *Figure 1* below this is described by the move from point G to point H.

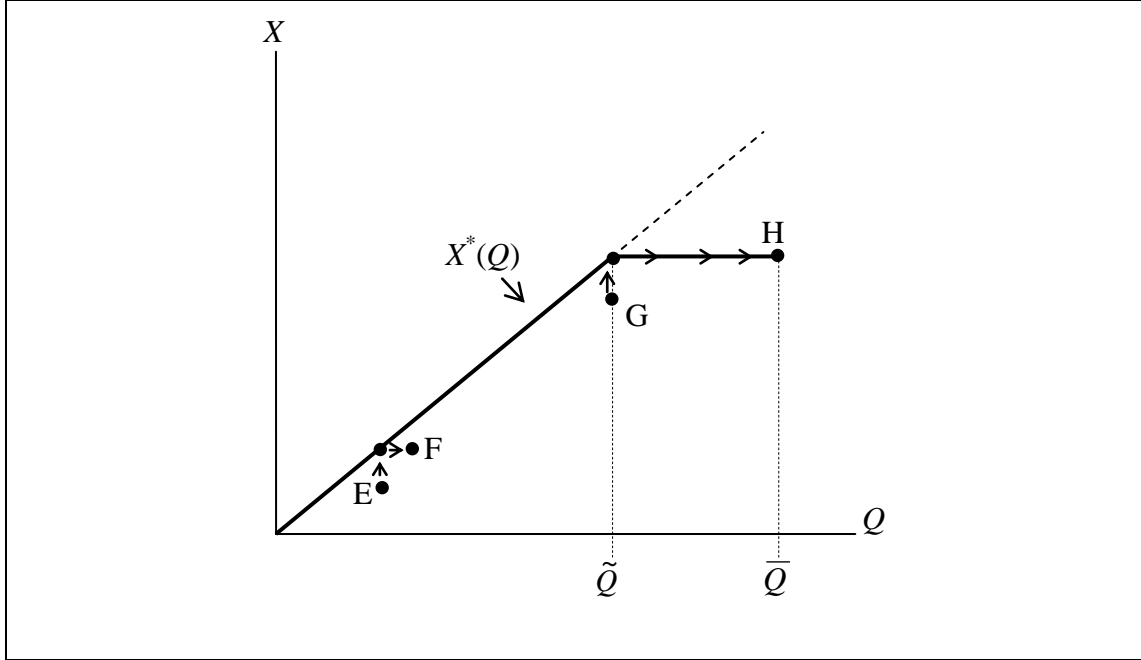


Figure 1: Investment dynamics.

3.1 Welfare in the presence of a cap on Q and free entry

In this sub-section, we find the function that describes the welfare associated with the considered market when a cap is imposed. Then, we maximize it with respect to \bar{Q} in order to find the socially optimal level of the cap. To better understand the analysis and its results we start with a short explanation of the different types of welfare losses induced by a cap. The first type of loss is induced by a cap that is set so low that units of good A that could potentially increase welfare are not produced. Similarly, we may have a welfare loss caused by a cap that is set so high that it enables the production of units that harms welfare. The third type of a welfare loss induced by a cap springs from the emergence of the run, as the expected contribution to welfare of each of the units added during the run is negative. The reason for that, as shown by (21), is that

the run occurs at a price level, $\bar{P} = \bar{X}/\bar{Q}$, that does not contain the wedge $\bar{\beta}$, balancing, as discussed above, the effect of the externality. In particular, by using (1) and (21), the expected present value of the welfare stream generated by each unit added during the run is $\frac{\bar{P}}{r-\mu} - \frac{M}{r} = (\lambda - 1) \cdot \frac{M}{r} < 0$, where the inequality follows from $\lambda < 1$.

After presenting the three types of welfare losses potentially induced by the cap, we now turn to the formal analysis.

Following the analysis presented in Section 2.2, the expected discounted social welfare in the presence of a cap and free entry is given by:

$$(24) \quad W(Q, X, \bar{Q}) = C(Q, \bar{Q}) \cdot X^\beta + \frac{X \cdot \ln(Q)}{r - \mu} - \frac{M \cdot Q}{r}.$$

Note that (24) is almost identical to (9), the function representing welfare in the case with no cap. The only difference is that now $W(Q, X, \bar{Q})$ and $C(Q, \bar{Q})$ are also functions of the size of the cap, \bar{Q} , and not merely functions of Q and X .

As follows from (24), maximizing the welfare function $W(Q, X, \bar{Q})$ with respect to \bar{Q} is equivalent to maximizing $C(Q, \bar{Q})$ with respect to the same variable.

Once set the focus on the maximization of $C(Q, \bar{Q})$, the following two scenarios must be considered:

- if \bar{Q} is set sufficiently close to the current level of Q , specifically, within the interval $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$, then $Q \geq \tilde{Q}$ and the next time the process X will hit the threshold \bar{X} , a run exhausting instantaneously the residual investment slots will be ignited;
- otherwise, if \bar{Q} is set sufficiently far from the current level of Q , specifically, within the interval $\bar{Q} > \bar{\beta} \cdot Q$, then $Q < \tilde{Q}$ and the next increments in Q will follow the sequential and incremental investment policy described in the section 2 as far as Q is below \tilde{Q} , while a run will start as soon as $Q = \tilde{Q}$.

Let's then proceed determining $C(Q, \bar{Q})$ in the two intervals of interest.

3.1.1 $C(Q, \bar{Q})$ when $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$

Within the interval, $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$, when the threshold \bar{X} is hit then a run brings Q to the cap \bar{Q} at once. Once reached the cap, no more changes in Q take place and welfare is given by the present value of the future flow of net surplus associated with a market quantity equal to \bar{Q} , i.e.,

$$(25) \quad w(Q, \bar{X}, \bar{Q}) = \frac{\bar{X} \cdot \ln(\bar{Q})}{r - \mu} - \frac{M \cdot \bar{Q}}{r},$$

Evaluating (24) at \bar{X} , comparing it to (25), applying (21) and simplifying, yields that in the range $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$ the function $C(Q, \bar{Q})$ is given by:

$$(26) \quad C(Q, \bar{Q}) = \bar{\beta}^\beta \cdot K \cdot \frac{\lambda \cdot \bar{Q} \cdot [\ln(\bar{Q}) - \ln(Q)] - (\bar{Q} - Q)}{\bar{Q}^\beta};$$

The following *Proposition 1* presents some of the properties of $C(Q, \bar{Q})$:

Proposition 1: Within the range $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$:

- i) $C(Q, \bar{Q})$ is a u-shaped function of \bar{Q} ;
- ii) At the left end of this range, $C(Q, \bar{Q})$ satisfies $C(Q, Q) = 0$;
- iii) At the right end of this range, $C(Q, \bar{Q})$ satisfies:

$$C(Q, \bar{\beta} \cdot Q) = \frac{K}{\beta - 1} \cdot \frac{\bar{\beta} \cdot \lambda \cdot g(\beta) - 1}{Q^{\beta-1}};$$

where $g(\beta) \equiv (\beta - 1) \cdot \ln(\bar{\beta})$ and satisfies $0 < g(\beta) < 1$ for any $\beta > 1$;

- iv) If $\bar{\beta} \cdot \lambda > \frac{1}{g(\beta)}$, then $C(Q, \bar{Q}) > 0$ at the right end of this range, and at a certain vicinity to its left, making $\bar{Q} = \bar{\beta} \cdot Q$ the maximizing level of \bar{Q} in that range;
- v) If $\bar{\beta} \cdot \lambda \leq \frac{1}{g(\beta)}$, then $C(Q, \bar{Q}) \leq 0$ throughout this range, making $\bar{Q} = Q$ the maximizing level of \bar{Q} in that range.

Proof: See appendix B. □

The logic behind part (ii) of Proposition 1 is that setting $\bar{Q} = Q$ stops any further investment and therefore $C(Q, \bar{Q})$, which represents the contribution of future entries, is, consistently, equal to 0.

Part (iv) shows that if, given the wedge $\bar{\beta} > 1$, the externality is relatively weak, i.e. λ is relatively high, then $C(Q, \bar{Q})$ is positive within a certain interval and reaches a maximum at $\bar{Q} = \bar{\beta} \cdot Q$. Note that setting the cap at this level implies that, in spite of the negative impact of the run, welfare is increasing in Q . Otherwise, as part (v) shows, in the presence of a relatively strong externality no welfare gains can be achieved by setting a \bar{Q} higher than the currently available market quantity Q .

3.1.2 $C(Q, \bar{Q})$ when $\bar{Q} > \bar{\beta} \cdot Q$

Within the interval $\bar{Q} > \bar{\beta} \cdot Q$, the initial value of Q is below \tilde{Q} . This means that, until the level \tilde{Q} has been reached, future changes in Q will be gradual and occur whenever X hits the threshold function $X^*(Q)$. The welfare analysis is therefore similar to the one, presented in Section 2.2, for the case where no cap has been introduced. In particular it will lead once again to (13), only with $C(Q, \bar{Q})$ at its LHS, instead of $C(Q)$. The integration constant, G is found using the boundary condition:

$$(25.1) \quad w(\tilde{Q}, \bar{X}, \bar{Q}) = \frac{\bar{X} \cdot \ln(\bar{Q})}{r - \mu} - \frac{M \cdot \bar{Q}}{r},$$

which is almost identical to (25) from the case in the previous sub-section. Evaluating both (24) and (25.1) at $Q = \tilde{Q}$, and equating them to one another yields a boundary expression for $C(\tilde{Q}, \bar{Q})$. Equating this expression to (13), evaluated at $Q = \tilde{Q}$, yields the integration constant G . Substituting G into (13) yields:

$$(27) \quad C(Q, \bar{Q}) = \frac{K}{\beta - 1} \cdot \left[\frac{\bar{\beta} \cdot \lambda - 1}{Q^{\beta-1}} - \frac{\bar{g}(\beta) \cdot \lambda \cdot \bar{\beta}^\beta}{\bar{Q}^{\beta-1}} \right],$$

where $\bar{g}(\beta) \equiv 1 - g(\beta)$ and satisfies $0 < \bar{g}(\beta) < 1$ for any $\beta > 1$. The following proposition shows some properties of $C(Q, \bar{Q})$ in this range:

Proposition 2: Within the range $\bar{Q} > \bar{\beta} \cdot Q$,

i) At the left end of this range, $C(Q, \bar{Q})$ satisfies:

$$C(Q, \bar{Q}) = \frac{K}{\beta - 1} \cdot \frac{\bar{\beta} \cdot \lambda \cdot g(\beta) - 1}{Q^{\beta-1}};$$

ii) $C_{\bar{Q}}(Q, \bar{Q}) > 0$;

iii) $\lim_{\bar{Q} \rightarrow \infty} C(Q, \bar{Q}) = \frac{K}{\beta - 1} \cdot \frac{\bar{\beta} \cdot \lambda - 1}{Q^{\beta-1}}.$

Proof: The three parts of the proposition follow directly from (27). □

Part (i) of the proposition shows continuity at $\bar{\beta} \cdot Q$ of the two branches of $C(Q, \bar{Q})$.

Part (iii) of the proposition presents the limit result for \bar{Q} going to infinity, a case which is equivalent to having no cap at all. Accordingly, $C(Q, \bar{Q})$ converges to the $C(Q)$ determined by (15) for the case where no cap was present. This implies that the condition for welfare-increasing market entries is $\bar{\beta} \cdot \lambda > 1$ in this case too.

3.1.3 $C(Q, \bar{Q})$ throughout its entire definition range

We illustrate the properties of the two branches of the function $C(Q, \bar{Q})$ in Figure 2.

Figure 2(a) refers to the case where $\lambda > \frac{1}{g(\beta)\bar{\beta}}$, Figure 2(b) to the case where

$\frac{1}{\bar{\beta}} < \lambda \leq \frac{1}{g(\beta)\bar{\beta}}$ while Figure 2(c) refers to the case where $\lambda \leq \frac{1}{\bar{\beta}}$.

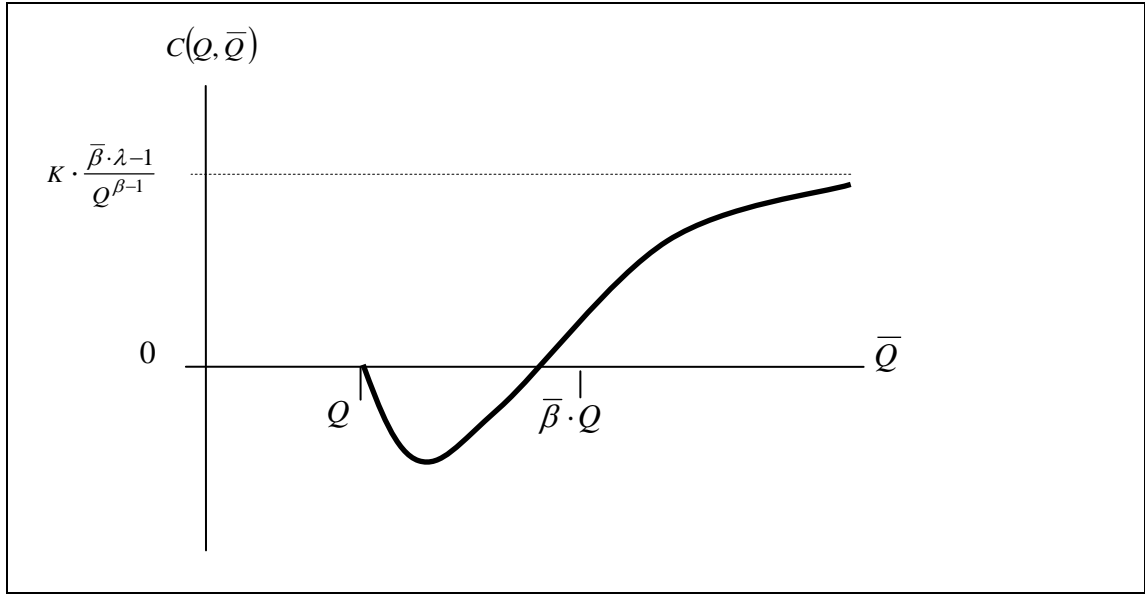


Figure 2(a): $C(Q, \bar{Q})$ for $\lambda > \frac{1}{g(\beta) \cdot \bar{\beta}}$.

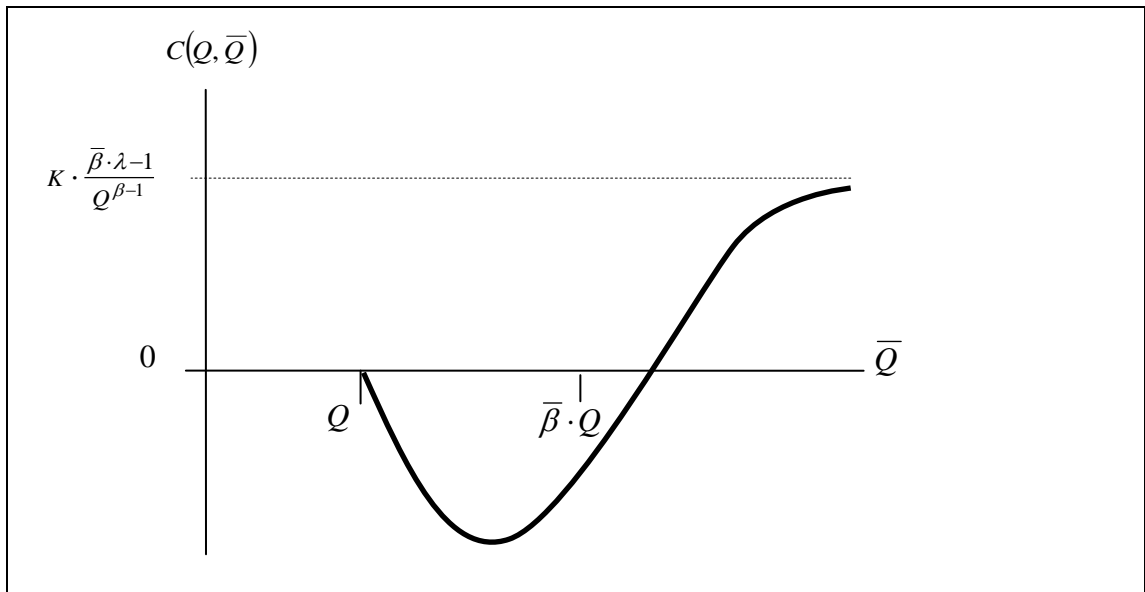


Figure 2(b): $C(Q, \bar{Q})$ for $\frac{1}{\bar{\beta}} < \lambda \leq \frac{1}{g(\beta) \cdot \bar{\beta}}$.

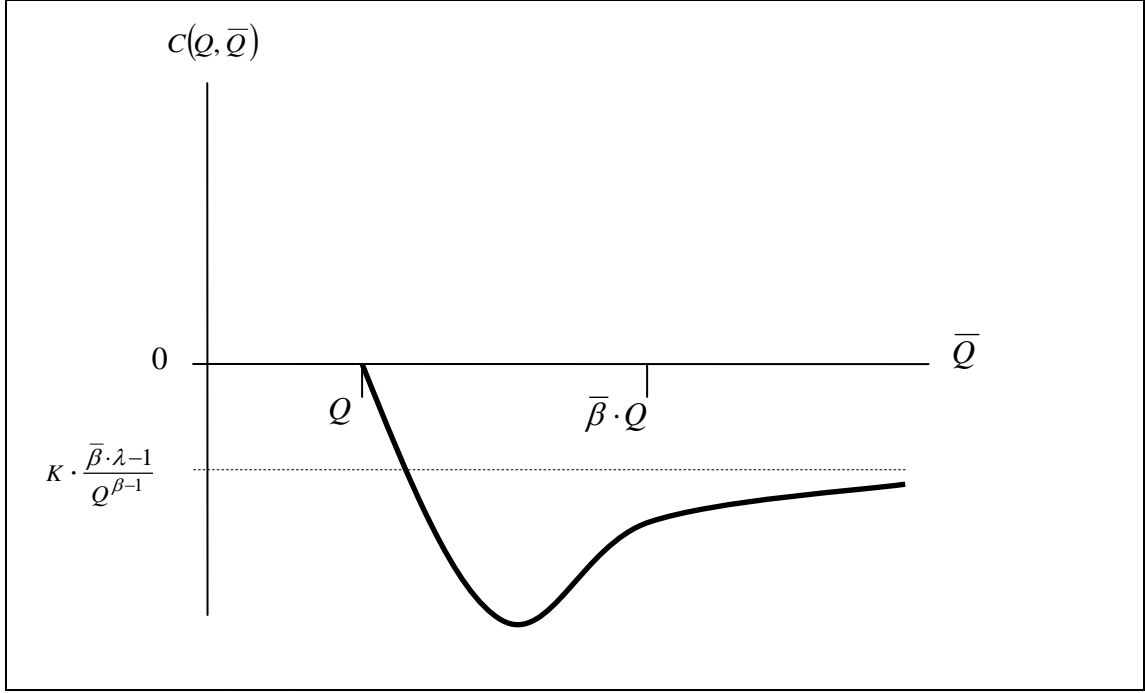


Figure 2(c): $C(Q, \bar{Q})$ for $\lambda < \frac{1}{\beta}$.

In all figures, we note that $C(Q, \bar{Q})$ equals 0 at the left end of the definition range for \bar{Q} , then the function falls to negative values as \bar{Q} rises, hits a minimum point within the range $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$ and rises with \bar{Q} from then onward. In both Figure 2(a) and Figure 2(b), the function $C(Q, \bar{Q})$ turns, as \bar{Q} rises, positive at a certain point. This occurs within the interval $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$ for the case where $\lambda > \frac{1}{g(\beta)\bar{\beta}}$ and within the interval $\bar{Q} > \bar{\beta} \cdot Q$ for the case where $\frac{1}{\beta} < \lambda \leq \frac{1}{g(\beta)\bar{\beta}}$. Differently, as shown in Figure 2(c), if $\lambda \leq \frac{1}{\beta}$ then, $C(Q, \bar{Q})$ is negative throughout its definition range.

Note that in all figures, $C(Q, \bar{Q})$ converges, as \bar{Q} tends to infinity, toward $C(Q)$, that is, the value associated with future market entries for the case where no cap is

present. This limit result is relevant for the definition of a welfare maximizing choice of \bar{Q} . The following Proposition summarizes these results about the optimal cap:

Proposition 3: If $\lambda \leq \frac{1}{\beta}$ then the cap should be set at the current level of Q , which implies allowing no further changes in Q . Otherwise, if $\lambda > \frac{1}{\beta}$, then it is optimal to push the cap to infinity, which actually implies having no cap at all. \square

By *Proposition 1* and *Proposition 2*, once the resulting welfare function is compared with the welfare function for the case with no cap, it is clear that, when $\lambda \leq \frac{1}{\beta}$, introducing a cap may reduce the welfare losses associated with future market entries. However, this is conditional on setting the cap sufficiently close to the current Q . Otherwise, the losses due to the occurrence of the run counterbalance the loss reduction due to introduction of the cap. These two competing effects lead to the non-monotonicity of $C(Q, \bar{Q})$ in the interval $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$. However, by the properties of $C(Q, \bar{Q})$, in the range of values of \bar{Q} securing a loss reduction, the closer the cap to the current Q , the lower the loss. Hence, setting the cap at the current market quantity Q maximizes welfare, as it implies no further losses, i.e. $C(Q, Q) = 0$. Note that this is equivalent to banning any further market entry.

In contrast, when $\lambda > \frac{1}{\beta}$, as welfare is, as discussed in Section 2.2, increasing in Q , introducing a cap would be detrimental for two reasons. First, as, due to the cap, it will deter from benefiting from further market entries which, as $\lambda > \frac{1}{\beta}$, would increase welfare. Second, as, during the run, market entries occur at a price below the

social marginal cost and their timing is suboptimal. Hence, setting a higher cap increases welfare as the production of further units at a price above the social production cost dominates the negative effect of the run, which, as \bar{Q} increases, takes place later in expected terms. These considerations then suggest that not having any cap at all is the policy that maximizes welfare.

4. Optimal cap with rationing

In this Section, we look at the case where entry licenses are distributed when the cap is announced. Each license is for an infinitesimally small increment of Q , i.e. ΔQ , and their number covers exactly the gap between the quantity currently available in the market and the cap.

Bartolini (1995) considered this case and determined the firms' optimal investment policy and the social welfare associated with the resulting market equilibrium. The cap level, introduced in order to reduce the impact of investment externalities, was exogenously set. In the welfare analysis, the losses associated with those externalities were not explicitly modeled and this was crucial for the conclusion that firms follow the first-best entry policy until the cap becomes binding. We consider Bartolini's definition of first-best partial. We then depart from his analysis by explicitly modeling the presence of an investment externality. We determine the firms' optimal investment policy in the presence of a cap and the resulting welfare and then, having welfare maximization as a target, we derive the optimal level of the cap.

We avoid the question of how the licenses were distributed, whether by auction, lottery, or any other mechanism. We merely assume that the distributing mechanism

adopted has no other implications except for providing each license owner with a right to invest at any time this owner wishes.⁸

As Bartolini (1995) shows, the competitive run of the free entry case does not emerge in the equilibrium of the licensing case. This is because firms holding a license do not fear that they will lose their entry option.

Absent the run, the analysis of the firm's optimal policy under licensing is very similar to the analysis in Section 2.1 for the case of no cap at all. The only difference is given by the presence of a cap binding at a certain point in time. We define, alongside the function $V(Q, X)$ that stands for the value of a firm once active in the market, the function $F(Q, X)$ that represents the value of a currently idle firm holding the right to enter that market. It is worth highlighting that in the case of free entry previously analyzed, this option was worthless, i.e. $F(Q, X) \equiv 0$, in that firms' entry was simply based on the zero-profit condition. In the current case, a no-arbitrage analysis, similar to one carried out in for $V(Q, X)$ in Appendix A, yields that:

$$(28) \quad F(Q, X) = H(Q) \cdot X^\beta,$$

where $H(Q)$ is to be found by imposing the following *Value Matching Condition*:

$$(29) \quad V[Q, X^*(Q)] = F[Q, X^*(Q)]$$

⁸ See Bartolini (1995) for a discussion of how alternative mechanisms impact on surplus extraction.

and the following *Smooth Pasting Condition*:

$$(30) \quad V_X[Q, X^*(Q)] = F_X[Q, X^*(Q)].$$

Note that conditions (29) and (30) stand in place of conditions (6) and (7) in Section 2.1 which were specific to the case of free entry where, as mentioned above, $F(Q, X) \equiv 0$. Despite this small difference from the analysis in Section 2.1, it can be easily shown that, once substituted (4) and (28) in (29) and (30), the investment threshold function in the licensing case is, $X^*(Q)$, as given by (8).

Alongside $X^*(Q)$, the solution of the system [29-30] yields an expression for $H(Q) - Y(Q)$ from which $F(Q, X)$ can be found once $V(Q, X)$ is determined using (8), (16) and (20).

As in the case of a cap under free entry, welfare is given by (24) and the analysis is similar to that proposed in Section 2.2, until (13) is obtained, with $C(Q, \bar{Q})$ at its LHS. Then $C(Q, \bar{Q})$ can be found using the following boundary condition:

$$(31) \quad C(\bar{Q}, \bar{Q}) = 0.$$

Condition (31) simply states that when Q is at \bar{Q} no more changes in Q are going to take place, and therefore the term $C(\bar{Q}, \bar{Q}) \cdot X^\beta$, which represents the value of such changes, should equal zero.

By applying (13) in (31), we can determine the integration coefficient G . Then, substituting this coefficient in (13) and simplifying yields:

$$(32) \quad C(Q, \bar{Q}) = \frac{K}{\beta - 1} \cdot (\bar{\beta} \cdot \lambda - 1) \cdot \left(\frac{1}{Q^{\beta-1}} - \frac{1}{\bar{Q}^{\beta-1}} \right).$$

From (32) it follows that $C(Q, \bar{Q})$, and, by (24) also welfare, is a monotonic function of \bar{Q} which is either increasing or decreasing in \bar{Q} , depending on the sign of $\bar{\beta} \cdot \lambda - 1$. We illustrate these two cases in Figure 3(a) and Figure 3(b), respectively.

Both figures show that $C(Q, \bar{Q})$ equals zero when \bar{Q} is set at the current level of Q and converges to $K \cdot \frac{\bar{\beta} \cdot \lambda - 1}{Q^{\beta-1}}$ as \bar{Q} goes to infinity. The main difference is about this limit being positive or negative and, consequently, about $C(Q, \bar{Q})$ rising or falling when moving towards it.

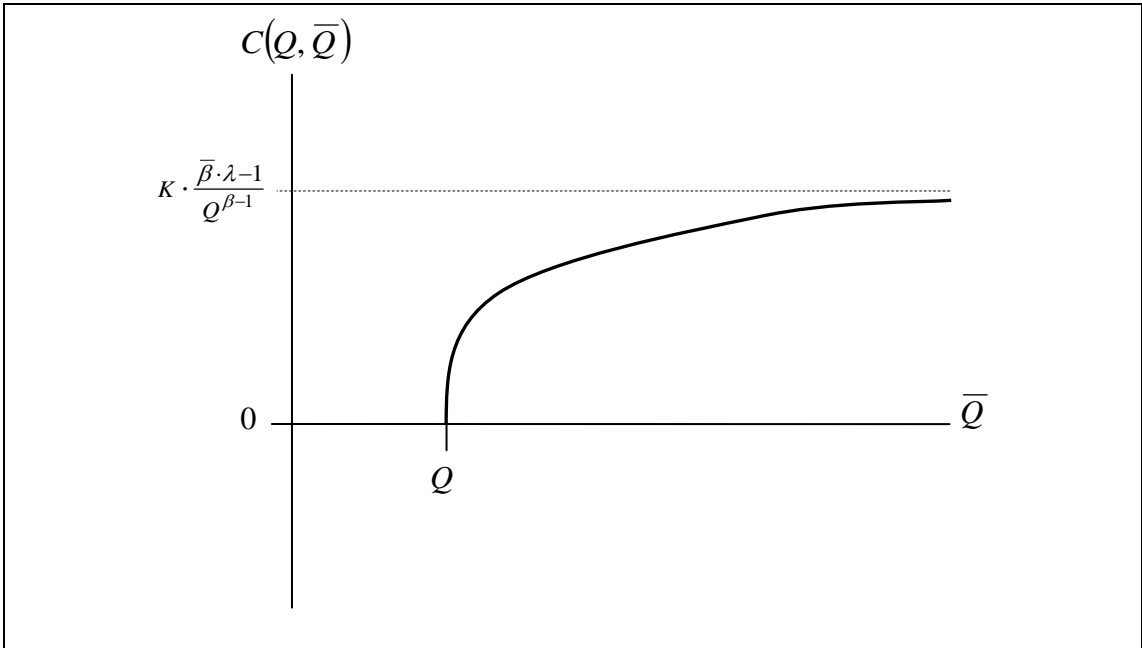


Figure 3(a): $C(Q, \bar{Q})$ when $\lambda > \frac{1}{\bar{\beta}}$.

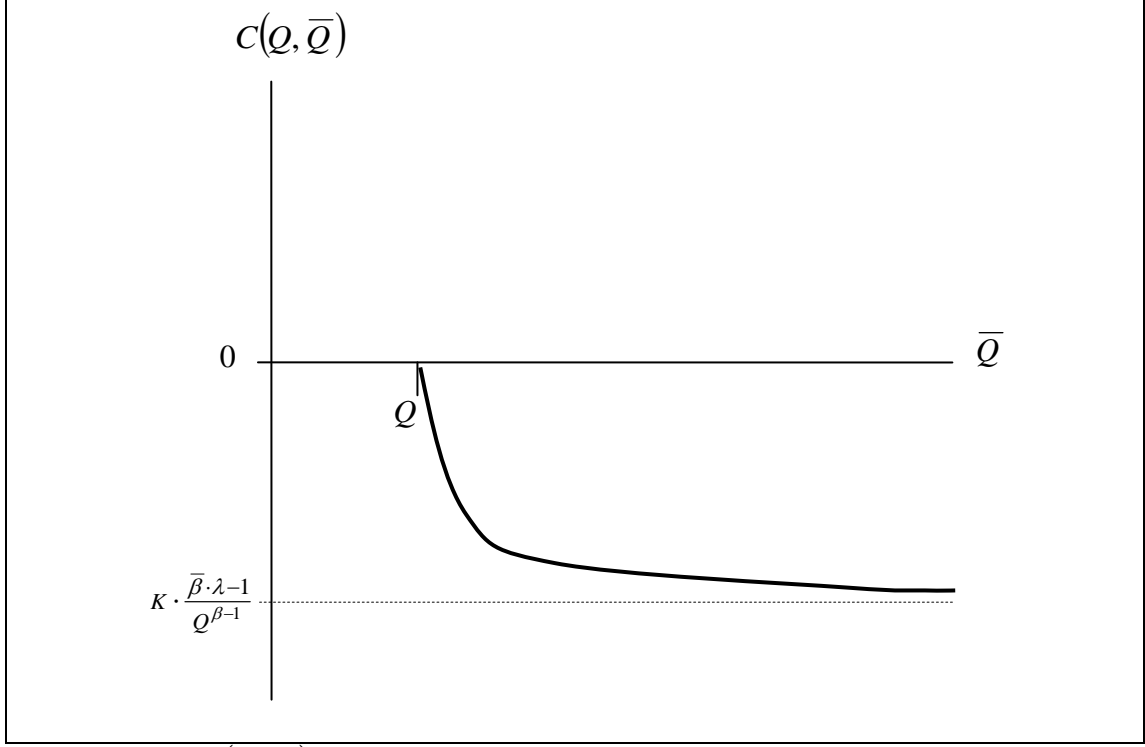


Figure 3(b): $C(Q, \bar{Q})$ when $\lambda < \frac{1}{\beta}$.

From (32) we can conclude that, when $\lambda \leq \frac{1}{\beta}$, the function $C(Q, \bar{Q})$ is decreasing in \bar{Q} , which means that introducing a cap may reduce welfare losses associated with future market entries. Further, by (32), the closer the cap to the current Q , the lower the loss. Hence, welfare is maximized if the cap is set equal to the current level of Q , which, as mentioned above, is equivalent to banning any further market entry. In contrast, when $\lambda > \frac{1}{\beta}$, the function $C(Q, \bar{Q})$ is increasing in \bar{Q} , because as welfare is increasing in Q , introducing a cap would deter from benefiting, once the cap becomes binding, from further welfare gains. Therefore, it is optimal to push \bar{Q} to infinity, i.e., having no cap at all.

Therefore, also in the case of rationing we come to the same policy recommendation given in the case of free entry. The main difference between the two cases concerns

the magnitude of losses that, in the case of free entry, include also the losses due to the occurrence of the run. In fact, comparing the impact of a cap on welfare under the two regimes, we conclude that:

Proposition 4: For any level of \bar{Q} exceeding the current market quantity Q , welfare under rationing is always larger than welfare under free entry.

Proof: See Appendix C. □

5. Timing of entry and welfare: public goals and private action

We start this Section with a numerical exercise illustrating the impact that the introduction of a cap has in terms of welfare under free entry and rationing. The resulting welfare levels will then be compared with the welfare levels accruing when, depending on whether $\lambda \leq \frac{1}{\beta}$ or $\lambda > \frac{1}{\beta}$, the welfare maximizing policies that should apply are, as stated in Proposition 1 and 2, a ban or no cap at all, respectively. In this exercise the relevant parameter values are as follows: $r = 0.1$, $M = 10$, $\lambda = 0.5$, $\mu = 0.02$ and σ takes values $\{0.1, 0.2, 0.3\}$. Finally, we consider two cap levels, $\bar{Q} = 25$ and $\bar{Q} = 50$ and we assume that the current level of X is equal to 50, i.e. $X_0 = 50$. Note that welfare levels under the different scenarios will be calculated on the basis of the number of active firms, Q , consistent with the current level of X and the investment threshold $X^*(Q)$.

<i>No policy</i>			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
<i>Welfare levels</i>	-112.05	205.53	553.91
<i>Optimal Policy</i>			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\beta \cdot \lambda$	0.73	0.91	1.13
	Ban	Ban	No cap
<i>Welfare levels</i>	70.76	303.09	553.91
<i>Cap and free entry</i>			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
<i>Welfare levels, $\bar{Q} = 25$</i>	-210.51	-64.76	-1.33
<i>Welfare levels, $\bar{Q} = 50$</i>	-133.23	90.79	233.71
<i>Cap and rationing</i>			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
<i>Welfare levels, $\bar{Q} = 25$</i>	-18.61	256.84	459.37
<i>Welfare levels, $\bar{Q} = 50$</i>	-91.95	227.32	499.39

Table 2: Welfare: optimal policy vs. cap.

Figures in Table 2 show the actual impact that introducing a cap has on the welfare associated to the market for good A. This impact, if compared with the welfare levels accruing when the optimal policies that our analysis suggests are implemented, is negative and relevant in magnitude. The cap induces always higher loss under free entry. This is due, as explained above, to the run that would characterize part of the market entry process. The loss due to the run lower as volatility increases. This is due to the probability of hitting the investment threshold being decreasing in the volatility level. This in turn slows down the entry process and pushes far in time the loss associated with the run. Hence, this loss, once discounted back, has a lower weight in present terms. Similarly, when raising the cap from 25 to 50, losses are lower in that the run will start, *ceteris paribus*, at a later time in expected terms. Under rationing, as the run is absent, welfare levels increase. We notice that whenever a ban should be announced, welfare is decreasing in the level of the cap. This is because, of course,

we are allowing for undesirable entries. In contrast, when capping investment is not optimal, having a higher cap allows for entries which positively contribute to welfare. We conclude the Section discussing the sub-optimality of private firms' market entry from the perspective of Society. The timing of entry set by a private firm, as presented in *Proposition 1*, abstracts from the consideration of the actual cost paid by the Society for the production of good A. In this respect, a natural benchmark is represented by the optimal entry policy that would be set by a planner maximizing the consumer surplus associated to Q units of good A net of their social cost, $M \cdot Q$, i.e.⁹

$$(33) \quad X^s(Q) = \bar{\beta} \cdot (r - \mu) \cdot \frac{M}{r} \cdot Q = \frac{X^*(Q)}{\lambda} > X^*(Q),$$

or, if expressed in terms of price level,

$$(34) \quad P^s = \frac{X^s(Q)}{Q} = \bar{\beta} \cdot (r - \mu) \cdot \frac{M}{r} = \frac{P^*}{\lambda} > P^*.$$

As the analysis in this section has shown, when entry rights are rationed, the imposition of a cap does not modify at all the timing of entry set by the potential entrants. Firms internalize only the portion λ of the social marginal cost of producing good A and keep entering the market following the entry policy that would be set in the absence of a cap, i.e. (8). In terms of expected entry timing, this means that the entry process is faster than socially desirable. The same considerations apply for the case of free entry too. In fact, before the run starts, i.e. for $Q < \tilde{Q}$, the firm's entry

⁹ See for instance Dixit and Pindyck (1994, Ch. 9).

policy is the same as in the case of rationing. Then, as soon as market quantity Q reaches the level \tilde{Q} , the entry process becomes even faster as the firms start running filling up instantaneously the remaining slots.

We illustrate this discussion by plotting in Figure 4 the investment thresholds dictating the entry timing in the first-best scenario and in the presence of a cap.

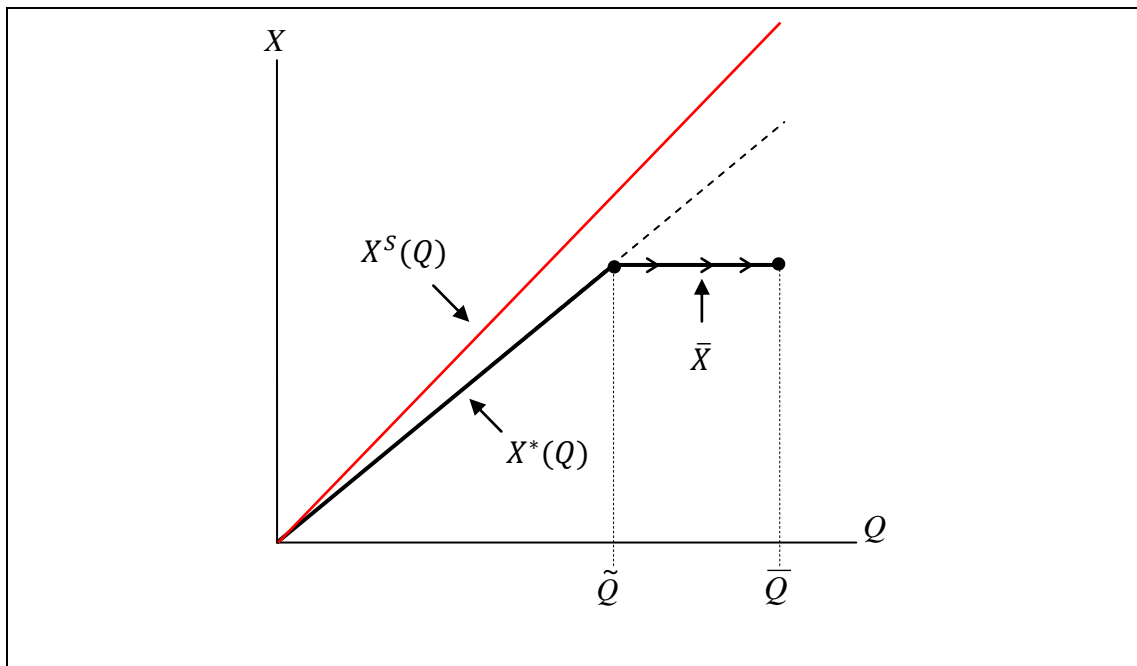


Figure 4: Investment thresholds: private and social perspectives.

Last, in order to provide a measure for the sub-optimality of the firms' entry timing from a social perspective, we compute, using the first-best investment threshold as benchmark, by how much time, in expected terms, the entry decision is sped up when a cap is introduced.

Let $\tau = \inf (t \geq 0 | P = \hat{P})$ denote the first time in which the random price P , moving from the initial level $P_0 < \hat{P}$, hits the generic upper barrier \hat{P} . By standard properties of the Geometric Brownian motion (see for instance Dixit ,1993), if $\mu > \sigma^2 / 2$ then:

$$(35) \quad E(\tau; P_0, \hat{P}) = \frac{\ln(\hat{P}/P_0)}{\mu - \frac{\sigma^2}{2}}$$

Otherwise, if $\mu \leq \sigma^2 / 2$, then the expected first hitting time goes to ∞ , implying that no market entries should be expected. In the following we focus therefore on the case where $\mu > \sigma^2 / 2$.

As mentioned above, firms enter the market earlier than socially desirable. Using (35), we can easily determine by how many time-periods entry is sped up. This corresponds to the expected first time the price P , moving from the level P^* , hits the barrier P^S , that is

$$(36) \quad E(\tau; P^*, P^S) = \frac{\ln(P^S / P^*)}{\mu - \sigma^2 / 2} = \frac{\ln(1/\lambda)}{\mu - \sigma^2 / 2}.$$

Similarly, once the run has been ignited, entry is sped up by the following amount of time:

$$(37) \quad E(\tau; \bar{P}, P^S) = \frac{\ln(P^S / \bar{P})}{\mu - \sigma^2 / 2} = \frac{\ln(\bar{\beta} / \lambda)}{\mu - \sigma^2 / 2}.$$

Now, for the sake of illustration, we propose a numerical exercise where we set $r = 0.1$ and $M = 10$, and let μ , σ and λ take values $\{0.01, 0.02, 0.03\}$, $\{0.1, 0.2, 0.3\}$ and $\{0.75, 0.50, 0.25\}$, respectively.

In table 3, we find the figures relative to the amount of time by which firms speed up their entry when entry rights are licensed or, in the case of free entry, before the run starts. The empty cells stand for combinations of μ and σ in which $\mu \leq \sigma^2 / 2$ and no market entries should be expected. We notice that, as expected, the higher the externality, the higher the time advance. The advance lowers as the drift increases. This is because the higher the drift, the faster price increases and, consequently, the higher the likelihood of hitting the upper barrier in a shorter amount of time.¹⁰ The opposite occurs when the volatility increases as, being lower the likelihood of hitting the barrier, a longer amount of time is needed before hitting the barrier.

$E(\tau; P^*, P^S)$			
$\lambda = 0.75$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	57.536	-	-
$\mu = 0.02$	19.179	-	-
$\mu = 0.03$	11.507	28.768	-
$\lambda = 0.50$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	138.629	-	-
$\mu = 0.02$	46.210	-	-
$\mu = 0.03$	27.726	69.315	-
$\lambda = 0.25$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	277.259	-	-
$\mu = 0.02$	92.420	-	-
$\mu = 0.03$	55.452	138.629	-

Table 3: The advancement of market entry timing under rationing and free entry (before the run).

¹⁰ See Dixit (1993, Section 6.1) for the derivation of the probability of first hitting.

Similar considerations hold when looking in table 4 at the time advance by which firms speed up their entry during the run. The only difference is that, being absent the consideration of the option value, the advance is substantially higher in relative terms.

$E(\tau; \bar{P}, P^S)$			
$\lambda = 0.75$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	115.073	-	-
$\mu = 0.02$	43.999	-	-
$\mu = 0.03$	30.705	98.083	-
$\lambda = 0.50$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	196.166	-	-
$\mu = 0.02$	71.030	-	-
$\mu = 0.03$	46.923	138.629	-
$\lambda = 0.25$			
	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
$\mu = 0.01$	334.795	-	-
$\mu = 0.02$	117.239	-	-
$\mu = 0.03$	74.649	207.944	-

Table 4: The advancement of market entry timing during the run.

6. Conclusion

In this paper, we have considered the problem of firms entering a market where output prices evolve randomly following a Geometrical Brownian motion and production has an adverse externality for Society. We have then studied the opportunity of introducing a cap on market entries in order to limit the welfare losses associated with this externality. This has been done considering both a scenario where firms may freely enter the market and a scenario where the right to enter the market is rationed by distributing licenses when the cap is announced. Once determined the consequent investment policy set by private firms, we have focused on

the implications that the imposed cap has on social welfare and raised a simple question: which is that cap level that should be set in order to maximize welfare? We have showed that, irrespective of the way by which the right to enter is allocated, the planner should set the cap at the currently existing quantity, i.e. ban further market entries, or have no cap at all. The policy to be chosen will depend on the strength of the option value associated with the entry decision. When its impact on the investment trigger dominates the impact of the externality, having no cap at all is welfare-maximizing; otherwise, banning further market entries is preferable. This result is relevant in that it implies that the justification of a cap on the aggregate market quantity based on social welfare considerations is not plausible. This means that the introduction of caps in the reality results from the consideration of objectives other than the actual social welfare. This may be the case when, for instance, political parties in the office opportunistically favor specific parts of the Society in order to increase the chances of conserving power. In this respect, our model allows identifying the cost of this choice for Society as a whole. Last, our analysis clearly shows that the introduction of a cap does not influence at all the firm's entry policy. This remains sub-optimal from a first-best perspective as firms keep not accounting, when entering the market, for the externality produced. This result opens to further research investigating whether, having the first-best in mind, considering a fee to be levied when entering the market may allow getting closer to the target.

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Appendix A

In this appendix we show that (4) presents the general form of the function $V(Q, X)$. For that, we use the standard no-arbitrage analysis of the literature on irreversible investment under uncertainty (see, for example, Dixit, 1989).

We start with the no-arbitrage condition:

$$(A.1) \quad r \cdot V(Q, X) \cdot dt = \frac{X}{Q} - \lambda \cdot M + E[dV(Q, X)],$$

which states that the instantaneous profit, $\frac{X}{Q} - \lambda \cdot M$, plus $E[dV(Q, X)]$, which is the expected instantaneous capital gain associated with a change in X , must equal the instantaneous normal return, $r \cdot V(Q, X) \cdot dt$.

Expanding $dV(Q, X)$ according to Ito's lemma, taking the expectancy using (3), and rearranging, yields:

$$(A.2) \quad \frac{E[dV(Q, X)]}{dt} = \frac{1}{2} \cdot \sigma^2 \cdot X^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X)$$

Applying (A.2) in (A.1) yields:

$$(A.3) \quad \frac{1}{2} \cdot \sigma^2 \cdot X^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X) - r \cdot V(Q, X) + \frac{X}{Q} - \lambda \cdot M = 0$$

Trying a solution of the type X^b for the homogenous part of this differential equation and then a linear form as particular solution of the entire equation, yields:

$$(A.4) \quad V(Q, X) = Z(Q) \cdot X^\alpha + Y(Q) \cdot X^\beta + \frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r},$$

where α and β are the roots of the following quadratic equation:

$$(A.5) \quad \frac{1}{2} \cdot \sigma^2 \cdot x^2 + \left(\mu - \frac{1}{2} \cdot \sigma^2 \right) \cdot x - r = 0.$$

The LHS is a quadratic function with a minimum point (since $\sigma^2 > 0$) and negative values at $x = 0$ and then $x = 1$ at the LHS due to the assumption that $r > \mu$. Based on that, $\beta > 1$ and $\alpha < 0$.

The term $\frac{X}{Q \cdot (r - \mu)} - \frac{\lambda \cdot M}{r}$ represents the expected value of the flow of profits if Q remains forever at its current level. The two other elements on the RHS of (A.4) represent therefore how the changes in Q over time, due to future market entries, are expected to affect the value of the firm.

By properties of the Geometric Brownian Motion, when X goes to 0 the probability of ever hitting $X^*(Q)$, and, consequently, Q changing, converges to 0. This implies that:

$$(A.6) \quad \lim_{X \rightarrow 0} [Z(Q) \cdot X^\alpha + Y(Q) \cdot X^\beta] = 0,$$

which in turn, leads, as $\alpha < 0$, to $Z(Q) = 0$, and therefore to the function present in (4).

Appendix B

By (26), at the left end of the considered range of values, i.e. when $\bar{Q} = Q$, we have:

$$(B.1) \quad C(Q, Q) = 0,$$

while at the right end, i.e. when $\bar{Q} = \bar{\beta} \cdot Q$,

$$(B.2) \quad C(Q, \bar{\beta} \cdot Q) = \frac{K}{\beta - 1} \cdot \frac{\bar{\beta} \cdot \lambda \cdot g(\beta) - 1}{Q^{\beta-1}},$$

which proves parts (ii) and (iii) of the proposition. To prove part (i), which states that $C(Q, \bar{Q})$ is a u-shape function of \bar{Q} within the range $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$, we start by differentiating $C(Q, \bar{Q})$, as given by (26), with respect to \bar{Q} , which yields:

$$(B.3) \quad C_{\bar{Q}}(Q, \bar{Q}) = \bar{\beta}^{\beta} \cdot K \cdot \frac{f(\bar{Q})}{\bar{Q}^{\beta+1}},$$

where:

$$(B.4) \quad f(\bar{Q}) \equiv -(\beta - 1) \cdot \lambda \cdot \bar{Q} \cdot [\ln(\bar{Q}) - \ln(Q)] + (\lambda - 1 + \beta) \cdot \bar{Q} - Q \cdot \beta.$$

From (B.3) it follows that the sign of $C_{\bar{Q}}(Q, \bar{Q})$ is the sign of $f(\bar{Q})$. Hence, to prove part (i) of Proposition 1 it suffices showing that: $f(\bar{Q}) < 0$ at the left end of this range, $f(\bar{Q}) > 0$ at its right end, and $f'(\bar{Q}) > 0$ throughout it.

To prove that, we start at the left end of this range. Applying $\bar{Q} = Q$ in (B.4) yields:

$$(B.5) \quad f(Q) = -(1 - \lambda) \cdot Q < 0,$$

where the inequality follows from $\lambda < 1$.

At the right end of this range, applying $\bar{Q} = \bar{\beta} \cdot Q$ in (B.4) yields:

$$(B.6) \quad f(\bar{\beta} \cdot Q) = [1 - g(\beta)] \cdot \bar{\beta} \cdot \lambda \cdot Q > 0,$$

where the inequality follows from $0 < g(\beta) < 1$ which is established in appendix D.

Finally, we have:

$$(B.7) \quad \begin{aligned} f'(\bar{Q}) &= \lambda \cdot \left[1 - (\beta - 1) \cdot \ln\left(\frac{\bar{Q}}{Q}\right) \right] + (\beta - 1) \cdot (1 - \lambda) \\ &> \lambda \cdot [1 - (\beta - 1) \cdot \ln(\bar{\beta})] = \lambda \cdot [1 - g(\beta)] > 0, \end{aligned}$$

The first equality follows from differentiating $f(\bar{Q})$, as given by (B.4), with respect to \bar{Q} . The first inequality follows from $\bar{Q} \leq \bar{\beta} \cdot Q$, $\beta > 1$ and $0 < \lambda < 1$, and the second inequality follows from $0 < g(\beta) < 1$. This concludes the proof of part (i).

Parts (iv) and (v) follow directly from parts (i), (ii) and (iii).

Appendix C

In this appendix we prove *Proposition 4* which states that welfare under licensing is larger than welfare under free entry for any $\bar{Q} > Q$.

From the general form of the welfare function, given by (24), which is relevant in both cases, it is clear that comparing welfare in the two cases reduces to the comparison of the value associated to $C(Q, \bar{Q})$ in the two cases. For the purpose of this appendix, in the case of free entry, we denote this function by $C^{FE}(Q, \bar{Q})$ and in the case of licensing, we denote it by $C^R(Q, \bar{Q})$. Under these notations, to prove the proposition it is sufficient to show that $C^R(Q, \bar{Q}) > C^{FE}(Q, \bar{Q})$, for each level of \bar{Q} . To do so we define the function:

$$(C.1) \quad D(\bar{Q}) \equiv C^R(Q, \bar{Q}) - C^{FE}(Q, \bar{Q}),$$

and prove that $D(\bar{Q}) > 0$ for any level of the cap, \bar{Q} . First, we will show it for the range $\bar{Q} > \bar{\beta} \cdot Q$. Then we will show that it also holds within the range $Q \leq \bar{Q} \leq \bar{\beta} \cdot Q$.

In the range $\bar{Q} > \bar{\beta} \cdot Q$, applying (32) for $C^R(Q, \bar{Q})$ and (28) for $C^{FE}(Q, \bar{Q})$ in (C.1) and simplifying yields:

$$(C.2) \quad D(\bar{Q}) = \frac{K}{\bar{Q}^{\beta-1}} \cdot [\lambda \cdot u(\beta) + 1] > 0,$$

where:

$$(C.3) \quad u(\beta) \equiv \bar{g}(\beta) \cdot \bar{\beta}^\beta - \bar{\beta}.$$

The inequality in (C.2) follows from $0 < \lambda < 1$ taken together with $u(\beta) > -1$ which is established in Appendix E.

To show that $D(\bar{Q}) > 0$ also in the range $Q < \bar{Q} < \bar{\beta} \cdot Q$, we return to (D.1) and now we apply (26) in it for $C^{FE}(Q, \bar{Q})$, together with, once again, (32) for $C^R(Q, \bar{Q})$. From (26), (32) and (C.1) it immediately follows that when the cap, \bar{Q} , is at its lowest possible level, i.e., at the current level of Q :

$$(C.4) \quad D(Q) = 0.$$

In addition, by continuity, and since it was already established that $D(\bar{Q}) > 0$ in the range $\bar{Q} > \bar{\beta} \cdot Q$:

$$(C.5) \quad D(\bar{\beta} \cdot Q) > 0.$$

Thus, $D(\bar{Q})$ equals zero at the left end of the range $Q < \bar{Q} < \bar{\beta} \cdot Q$ and it is strictly positive at the right end of this range. Therefore, the only manner by which $D(\bar{Q})$ can be negative in some sub-part of this range is that within that sub-part it has a local minimum point, i.e., a point in which $D'(\bar{Q}) = 0$ and $D''(\bar{Q}) > 0$. However, this is not possible because within this range, if $D'(\bar{Q}) = 0$ at a certain point then $D''(\bar{Q}) > 0$ must also hold at that point. To show this, we return to (C.1), apply (26) and (32) in it, differentiate and simplify. This yields that within the range $Q < \bar{Q} < \bar{\beta} \cdot Q$:

$$(C.6) \quad D'(\bar{Q}) = (\beta - 1) \cdot \left[\frac{\bar{\beta} \cdot \lambda - 1 - \bar{\beta}^\beta \cdot (\beta - 1 + \lambda)}{\bar{Q}^\beta} + \frac{\bar{\beta}^\beta \cdot (\beta - 1) \cdot \lambda \cdot \frac{\ln(\bar{Q}) - \ln(Q)}{\bar{Q}^\beta} + \frac{\bar{\beta}^\beta \cdot \beta \cdot Q}{\bar{Q}^{\beta+1}} \right]$$

Applying (C.6) in $D'(\bar{Q}) = 0$ and simplifying yields that when $D'(\bar{Q}) = 0$ holds the following equation must also hold:

$$(C.7) \quad \bar{\beta}^\beta \cdot (\beta - 1) \cdot \lambda \cdot [\ln(\bar{Q}) - \ln(Q)] = \bar{\beta}^\beta \cdot (\beta - 1 + \lambda) - (\bar{\beta} \cdot \lambda - 1) - \frac{\bar{\beta}^\beta \cdot \beta \cdot Q}{\bar{Q}}.$$

Returning to (C.6), differentiating again, and simplifying, yields:

$$(C.8) \quad D''(\bar{Q}) = \frac{\beta \cdot (\beta - 1)}{\bar{Q}^{\beta+1}} \cdot \left\{ \begin{aligned} & -(\bar{\beta} \cdot \lambda - 1) + \bar{\beta}^\beta \cdot (\beta - 1 + \lambda) \\ & + \bar{\beta}^{\beta-1} \cdot \lambda - \frac{\bar{\beta}^\beta \cdot (\beta + 1) \cdot Q}{\bar{Q}} \\ & - \bar{\beta}^\beta \cdot (\beta - 1) \cdot \lambda \cdot [\ln(\bar{Q}) - \ln(Q)] \end{aligned} \right\}.$$

Applying (C.7) in (C.8) and simplifying yields that within the range $Q < \bar{Q} < \bar{\beta} \cdot Q$,

when $D'(\bar{Q}) = 0$:

$$(C.9) \quad D''(\bar{Q}) = \frac{(\beta - 1) \cdot \beta}{\bar{Q}^{\beta+1}} \cdot \bar{\beta}^{\beta-1} \cdot \left(\lambda - \frac{\bar{\beta} \cdot Q}{\bar{Q}} \right) < \frac{(\beta - 1)^2}{\bar{Q}^{\beta+1}} \cdot \bar{\beta}^\beta \cdot (\lambda - 1) < 0,$$

where the first inequality follows from $\bar{Q} < \bar{\beta} \cdot Q$ and the second from $0 < \lambda < 1$. \square

Appendix D

In this appendix we prove that $g(\beta) > 0$ within the range $\beta > 1$. To do so we start by recalling that

$$(D.1) \quad \bar{\beta} \equiv \frac{\beta}{\beta - 1},$$

and therefore:

$$(D.2) \quad \frac{d\bar{\beta}}{d\beta} = \frac{1 \cdot (\beta - 1) - \beta \cdot 1}{(\beta - 1)^2} = \frac{-1}{(\beta - 1)^2}.$$

We use (D.1) and (D.2) to calculate the following limit:

$$(D.3) \quad \lim_{\beta \rightarrow \infty} g(\beta) = \lim_{\beta \rightarrow \infty} \left[\frac{\ln(\bar{\beta})}{\frac{1}{\beta-1}} \right] = \lim_{\beta \rightarrow \infty} \left[\frac{\frac{1}{\bar{\beta}} \cdot \frac{-1}{(\beta-1)^2}}{\frac{-1}{(\beta-1)^2}} \right] = \lim_{\beta \rightarrow \infty} \frac{1}{\bar{\beta}} = 1,$$

where the second equality follows from De L'Hopital's rule. Similarly, we also have:

$$(D.4) \quad \lim_{\beta \rightarrow 1} g(\beta) = \lim_{\beta \rightarrow 1} \frac{1}{\bar{\beta}} = 0,$$

Taking the first and second derivatives of $g(\beta)$ with respect to β yields:

$$(D.5) \quad g'(\beta) = \ln(\bar{\beta}) - \frac{1}{\beta},$$

$$(D.6) \quad g''(\beta) = \frac{-1}{\beta^2 \cdot (\beta-1)} < 0.$$

(D.5), together with (D.1), leads to:

$$(D.7) \quad \lim_{\beta \rightarrow \infty} g'(\beta) = 0,$$

Last, from (D.6) and (D.7) it follows that $g'(\beta) > 0$ for any $\beta > 1$. This result, taken together with (D.3) and (D.4) establishes that $0 < g(\beta) < 1$ for any $\beta > 1$.

Appendix E

Lemma 1: $u(\beta) > -1$ for all $\beta > 1$.

Proof: Applying (D.2) and rearranging terms reveals that $u(\beta) > -1$ is equivalent to:

$$(E.1) \quad \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot [1 - g(\beta)] > 1.$$

To show that inequality (E.1) holds we define its LHS by $h(\beta)$. The following two characteristics of $h(\beta)$ lead directly to $h(\beta) > 1$.

- (a) $\lim_{\beta \rightarrow 1} h(\beta) = 1$
- (b) $h'(\beta) > 0 \quad \forall \beta > 1.$

To prove (a): we calculate the following limits:

$$(E.2) \quad \begin{aligned} \lim_{\beta \rightarrow 1} (\beta-1)^{\beta-1} &= \lim_{\beta \rightarrow 1} e^{\ln(\beta-1)^{\beta-1}} = \lim_{\beta \rightarrow 1} e^{(\beta-1) \cdot \ln(\beta-1)} = \lim_{\beta \rightarrow 1} e^{\frac{\ln(\beta-1)}{\frac{1}{\beta-1}}} \\ &= \lim_{\beta \rightarrow 1} e^{\frac{\frac{1}{\beta-1}}{\frac{-1}{(\beta-1)^2}}} = \lim_{\beta \rightarrow 1} e^{-(\beta-1)} = 1 \end{aligned}$$

which, together with (D.4), $\beta > 1$, and the definition of $h(\beta)$ lead to (a).

The following derivative is useful for proving (b):

$$\begin{aligned}
 \text{(E.3)} \quad \frac{d \left[\frac{\beta^\beta}{(\beta-1)^{\beta-1}} \right]}{d\beta} &= \frac{de^{\ln \frac{\beta^\beta}{(\beta-1)^{\beta-1}}}}{d\beta} = \frac{de^{\beta \cdot \ln \beta - (\beta-1) \cdot \ln(\beta-1)}}{d\beta} \\
 &= e^{\beta \cdot \ln \beta - (\beta-1) \cdot \ln(\beta-1)} \cdot [\ln \beta + 1 - \ln(\beta-1)] \\
 &= \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot \ln(\bar{\beta})
 \end{aligned}$$

Using (E.3) when differentiating $h(\beta)$ yields:

$$\begin{aligned}
 \text{(E.4)} \quad h'(\beta) &= \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot \ln(\bar{\beta}) \cdot g(\beta) + \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot g'(\beta) \\
 &= \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot [\ln(\bar{\beta}) \cdot g(\beta) + g'(\beta)] \\
 &= \frac{\beta^\beta}{(\beta-1)^{\beta-1}} \cdot \left\{ \frac{1}{\beta} - (\beta-1) \cdot [\ln(\bar{\beta})]^2 \right\}.
 \end{aligned}$$

To prove (b) it now remains to show that the expression in the square brackets is positive. For that purpose, we define it as following:

$$\text{(E.5)} \quad \phi(\beta) \equiv \frac{1}{\beta} - (\beta-1) \cdot [\ln(\bar{\beta})]^2,$$

We calculate the following limit:

$$\begin{aligned}
 (E.6) \quad \lim_{\beta \rightarrow \infty} \phi(\beta) &= \lim_{\beta \rightarrow \infty} \left\{ 0 - \frac{[\ln(\bar{\beta})]^2}{\frac{1}{\beta-1}} \right\} = - \lim_{\beta \rightarrow \infty} \frac{2 \cdot [\ln(\bar{\beta})] \cdot \frac{1}{\bar{\beta}} \cdot \frac{-1}{(\beta-1)^2}}{\frac{-1}{(\beta-1)^2}} \\
 &= - \lim_{\beta \rightarrow \infty} \frac{2 \cdot \ln(\bar{\beta})}{\bar{\beta}} = 0,
 \end{aligned}$$

and the derivative:

$$\begin{aligned}
 (E.7) \quad \phi'(\beta) &= -\frac{1}{\beta^2} - 1 \cdot [\ln(\bar{\beta})]^2 - (\beta-1) \cdot 2 \cdot \ln(\bar{\beta}) \cdot \frac{1}{\bar{\beta}} \cdot \frac{-1}{(\beta-1)^2} \\
 &= -\frac{1}{\beta^2} - [\ln(\bar{\beta})]^2 + 2 \cdot \ln(\bar{\beta}) \cdot \frac{1}{\bar{\beta}} = -\left[\frac{1}{\beta} - \ln(\bar{\beta}) \right]^2 < 0.
 \end{aligned}$$

Inequality (E.7) and the limit (E.6) imply that $\phi(\beta) > 0$ and therefore, by (E.5) and (E.4), we have that $h'(\beta) > 0$, for any $\beta > 1$. This establishes (b).