Inflation and Welfare in a Competitive Search Equilibrium with Asymmetric Information*

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Abstract

We study an economy characterised by indivisible goods, competitive search and asymmetric information. Money is essential. Buyers decide their cash holdings after observing the contracts posted by firms and experience match-specific preference shocks which remain unknown to sellers. Firms are allowed to post general contracts specifying a price and a probability to trade, but the optimal contract implies a single price so that only those who value the good more than that price are able to trade. When the number of potential buyers is bounded, we show that the real price of the good is decreasing in the rate of inflation and, consequently, in the nominal rate of interest. Because of asymmetric information and indivisibility, monetary policy can exploit this relationship, and welfare is maximised away from the Friedman rule. The same result, but for a different reason, is obtained when there is buyers free entry. At the Friedman rule, asymmetric information causes a congestion effect, and an inflationary monetary policy can improve ex ante *welfare.*

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1 Introduction

One of the oldest issues in macroeconomics is whether some inflation can be good for the economy. Central banks in industrialised countries seem to think so, as they tend to set inflation targets around 2 percent, but there is very little theoretical support for this practice. As a matter of fact, neither models in the classical tradition, recently revived by the so-called *New Monetarist* approach, nor *New Keynesian* models are able to account for the optimality of a positive rate of inflation.[1] A noticeable exception is given by the literature, surveyed below, which, building on Bewley [4], takes seriously agents' heterogeneity and studies the possible redistributive effects of monetary policy.

In this paper, we revisit this issue and we propose an argument, which we believe has not yet been explored by the literature, that rationalises the existence of a long-run output inflation trade-off and the non-optimality of the Friedman rule. The argument does not rely on wage and price stickiness or irrational expectations, but rather on the ability of inflation to reduce the distortions caused by asymmetric information.

We consider a micro-founded model of money characterised by competitive search as in Rocheteau and Wright [31]. We study competitive search not only because is more tractable - it is easier to study asymmetric information under under price posting than under other trading protocols, such as bargaining - but also because it is more realistic: trade is usually organised by specialised agents, like stores, that can be easily located but have capacity constraints, so that agents typically face a trade-off between prices and probabilities. In this model, the good is indivisible[2] and buyers choose their money balances after observing the posted contracts, but before learning their types. Importantly, although in principle any sophisticated screening mechanism is allowed, the optimal mechanism turns out to be quite simple, with sellers posting a single price for all types of buyers; only those who value the good more than that price are able to trade.[3]

In essence, our argument runs as follows. Because of asymmetric information and bilateral matching, sellers face a downward sloping demand function, as in the textbook monopoly problem. When the number of buyers is bounded and all buyers enter the market, higher inflation induces sellers to lower their markups and this increases aggregate demand and output. An increase in inflation, up to some limit, affects positively the number of trades that occur in the economy without affecting buyers' participation, so that the optimal monetary policy is found away from the

[1] *New Monetarist* models typically find that the Friedman rule, which implies deflation, is optimal. Some exceptions are surveyed in section 2. Optimal monetary policy in the *New Keynesian* models has been recently reviewed by Schmitt-Grohé and Uribe [35]. Models that incorporate nominal rigidities as the sole source of monetary non-neutrality predict the the optimal rate of inflation is zero, since price stability eliminates the inefficiencies brought about by the presence of price adjustment cost.

[2] As we will argue later on, this assumption is equivalent to assume linear utility, divisible good, and capacity constraint.

[3] This is due to the fact that, also in our simple setting of monopolistic screening, the famous *Revenue Equivalence Theorem* of Myerson [30] holds.

Friedman rule. A positive rate of inflation is optimal also when there is free entry of buyers but, in this case, the mechanism is quite different. At the Friedman rule, the informational rent attracts too many buyers and this creates a congestion effect. It is optimal, then, for the monetary authority, to discourage participation, by increasing the cost of holding cash.

Similarly to what happens in *New Keynesian* models, markups of prices over costs play a central role in our analysis. However, while in those models, markups are exogenous and monetary policy affects output because of nominal rigidities, in our model markups are endogenous and adjust in response to monetary policy, thus producing real effects. Interestingly, although our paper is purely theoretical, the mechanism we highlight rationalises the apparent inverse correlation between inflation and the series of aggregate markups recently estimated by De Loecker and Eeckhout [9] for the US in the last fifty years.

The paper is organised as follows. In section 2 we briefly review the literature. In section 3, we present the basic environment. In sections 4 and 5, we derive value functions and optimal contracts. In section 6, we study an economy in which the number of buyers is limited while in section 7 we analyse the same economy with an unlimited number of buyers. Optimal monetary policy is derived for each case. Section 8 provides a general discussion of our results. Section 9 concludes. All proofs are in the Appendix.

2 Literature review

The model closest to ours is Masters [28] that also considers a market characterised by competitive search where buyers' match specific tastes are private information. Differently from our model, sellers produce goods of different quality and production occurs before the goods are retailed. Head and Kumar [20] propose a random matching model where, as in Burdett and Judd [7], incomplete information gives rise to price dispersion. In this model, the socially optimal rate of inflation may exceed that prescribed by the Friedman rule if buyers decide to observe more than one price.

There are other few micro-founded models of money, but without asymmetric information, in which the Friedman rule is not optimal. In an economy with heterogeneity in discounting and consumption risk, Boel and Camera [6] show that an inflationary monetary policy may be socially desirable when the government issues bonds that are sufficiently illiquid. Aruoba and Chung [2] find that the Friedman rule is non-optimal in a Rocheteau-Wright economy with physical capital. They show that, in contrast to standard Ramsey models, inflation is the most efficient way to tax rents associated to the use of money. Other papers find that deviating from the Friedman rule may be optimal when there are trading externalities (as in Shi [34]) or a congestion effect (as in Rocheteau and Wright [31], Craig and Rocheteau [8], and Geromichalos and Jung [16]).

Active monetary policies may be optimal in economies with a non-degenerate distribution of money. This literature, which follows the seminal paper of Bewley [4], has been mainly developed by Scheinkman and Weiss [33], Kehoe, Levine and

Woodford [22], Levine [25] and more recently by Wallace [36], Guerrieri and Lorenzoni [17] and Lippi, Ragni and Trachter [26]. In these models, where agents hold money to self-insure against uncertain trading prospects, a monetary expansion, inducing inflation, may benefit unlucky traders who ran out of cash.

The case of adverse selection with divisible goods in a Lagos-Wright economy has been analysed by Ennis [13], Faig and Jerez [14] and Dong and Jiang [11]. An analysis of competitive search with indivisible goods, symmetric information and single types can be found in Han, Julien, Petursdottir and Wang [19]. In all these models, however, the optimal monetary policy is usually found at the Friedman rule. Competitive search under asymmetric information, in a non-monetary model, has been first studied by Guerrieri, Shimer and Wright [18].

Our paper is also related to those few micro-founded models that study the trade-off between output and inflation. Rocheteau, Rupert, and Wright [32], for example, show that the relationship between anticipated inflation and unemployment needs not be zero, even in the long run, as predicted by the theory of the expectations-augmented Phillips curve. In their model, however, it is optimal to reduce inflation to a minimum. Dong [10] extend this analysis to a more general environment, and finds that, when agents' preferences are not restricted to be separable, the effect of inflation on aggregate unemployment becomes ambiguous. Deflation, however, is still optimal. Unlike these paper, we study a model where a permanent trade-off between output and inflation may occur, together with the non-optimality of the Friedman rule.

3 Environment

In a discrete-time economy, there exists a continuum of buyers of measure $\mathcal{B}$ and a continuum of sellers of measure $\mathcal{S}$, who live forever. Both buyers and sellers are risk neutral and *ex ante* homogenous. As in Lagos and Wright [24] and Rocheteau and Wright [31], each period is divided into two sub-periods. In the first sub-period, agents interact in a decentralised market (DM) where they meet bilaterally. In the second sub-period agents trade in a centralised frictionless Arrow-Debreu market (CM). In the CM both buyers and sellers consume a divisible good that anyone can produce. In the DM, only buyers consume a perishable indivisible good, which is produced by sellers, exactly in one unit at constant cost $c \geq 0$. The fact that in the DM sellers have no desire to consume during the day and buyers are not able to produce, generates a double coincidence problem. This, together with the assumption that agents are anonymous, gives rise to the need for a medium of exchange. In the DM, buyers derive utility θ from consuming the indivisible good. The instantaneous utility functions of buyers and sellers are given by:

$$U^b = \mathbb{I}\theta + X - H \tag{1}$$

$$U^s = -\mathbb{I}c + X - H \tag{2}$$

where $X \in \mathbb{R}_+$ is the consumption of the CM good, $H \in \mathbb{R}_+$ is the amount of labor supplied in the CM and $\mathbb{I}$ is an indicator function giving 1 if trade occurs and 0 otherwise. We assume that H produces X one-for-one. Good X is the numeraire in period t. Utility θ is a random variable representing the buyer's valuation of the good and identifies the buyer's *type*. We assume that θ is distributed according to a cumulative distribution function $F(\theta)$, and density $f(\theta) > 0$ on the support $\Theta \equiv [\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} \leq c$. Moreover, we assume that the cumulative distribution function follows the monotone hazard rate condition $d\left(f(\theta)/(1-F(\theta))\right)/d\theta \geq 0$.[4]

The economy is characterised by competitive search. In particular, we assume that all sellers simultaneously post contracts that specify the terms at which they are willing to trade. Contracts $\mathcal{C}$ are posted by sellers during the CM, before buyers chose their money holdings. Each seller commits to selling the indivisible DM good according to the posted contract. After observing all posted contracts, buyers direct their search towards the set of sellers posting the most attractive offers. The set of sellers posting the same offers and the set of buyers directing their search towards them form a submarket. In each submarket, agents meet randomly according to the matching function specified below. We assume that, in a given period, a seller can meet at most one buyer. Then, matching takes place and, for each match, the draw θ is realised and privately observed by the buyer. Next, the buyer can decide either to buy the good or to walk away. Although they never observe the buyer's type, sellers know $F(\theta)$.

Sellers and buyers are aware that their matching probabilities depend on the contract that they, respectively, post and seek. We assume that all sellers participate in the DM, at no cost, while only a subset of buyers $\overline{\mathcal{B}} \subseteq \mathcal{B}$ enter the DM, searching for $\mathcal{C}$, at each date. We will consider both the case in which $\overline{\mathcal{B}} = \mathcal{B}$ (*limited number of buyers*), and the case in which the set $\overline{\mathcal{B}}$ is unlimited (*buyers' free entry*) but buyers face an entry cost.

The mass of matches realised is given by a constant-return to scale matching function $s\left(\mathcal{S}, \overline{\mathcal{B}}\right)$. As in the standard search models, we are only interested in the ratio of buyers and sellers. Let now $n = \overline{\mathcal{B}}/\mathcal{S}$ denote the market tightness for the contract $\mathcal{C}$ and define the function $\sigma(n) \equiv s(n, 1)$. For simplicity, we normalise $\mathcal{S}$ to 1 and we set $\mathcal{B}/\mathcal{S} = N$. Then, a seller posting $\mathcal{C}$ meets a buyer with probability $\sigma(n)$ and a buyer meets a seller with probability $\sigma(n)/n$. We assume: $\sigma'(n) > 0$, $\sigma''(n) < 0$, $\sigma(0) = 0$, $\lim_{n\to\infty} \sigma(n) = 1$ and $\lim_{n\to 0} \sigma'(n) = 1$. By posting a more favourable contract, a seller attracts more buyers, which increases the seller contact rate $\sigma(n)$ and reduces the buyer contact rate $\sigma(n)/n$. In equilibrium buyers will be indifferent about where to apply, at least among posted contracts that attract some buyers.

Money is an intrinsically worthless object, perfectly divisible and storable in any non-negative quantity. Money supply grows at a constant gross rate μ, so that $M_{+1} = \mu M$. We assume that new money is injected (withdrawn if $\mu < 1$) through lump-sum transfers (taxes) in the CM. For the sake of simplicity, we assume that these transfers go only either to buyers or to sellers.[5] The size of the transfer is

[4] Economies that satisfies this property are sometimes called "regular".

[5] The model will be written assuming that the transfer goes to buyers. In section 7, where we assume that the number of potential buyers is unlimited, we consider the case in which transfers are given to sellers.

$T = (\mu - 1) M$. We restrict attention to policies where $\mu \geq \beta$, since it can easily be checked that for any $\mu < \beta$ there is no monetary equilibrium. Furthermore, when $\mu = \beta$ (i.e. at the Friedman rule), we only consider equilibria obtained by taking the limit $\mu \to \beta$.

The sequence of events is described in figure 1.

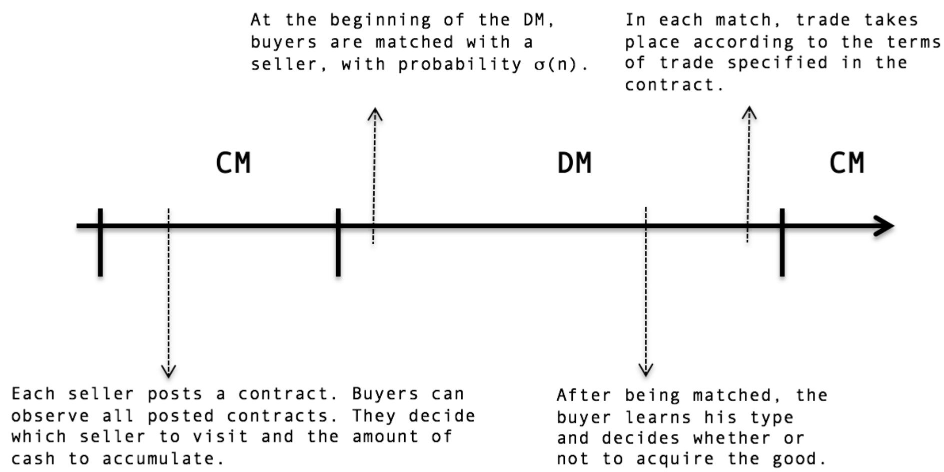

Figure 1: Time line

Our model combines perfect competition in the CM and competitive search in the DM. In equilibrium, agents maximise their utility, taking as given the sequence of prices in the Arrow-Debreu market and the sequence of conditions in the competitive search market, which will be derived in section 5. Contracts are optimally designed on the basis of rational expectations about the evolution over time of the CM prices and the DM equilibrium conditions.

4 Value functions

In the CM, we define $W^b(m)$ and $W^s(\tilde{m})$ to be the lifetime utility of a buyer and a seller, respectively. The value of being a buyer in the CM is given by:

$$W^b(m) = \max_{\{X,H,m_{+1}\}} \left[X - H + \beta V^b \left(m_{+1} \right) \right] \tag{3}$$

$$\text{s.t.} \qquad X + \phi m_{+1} = H + \phi \left(m + T \right) \tag{4}$$

where ϕm represents the buyer's real balances (in terms of CM good), ϕT is the real monetary transfer (or tax) that a buyer receives, and V^b is the value of being a buyer in the DM. Substituting (4) into (3), the buyer's value function in the CM can be written as:

$$W^b(m) = \phi(m+T) + \max_{\{m_{+1}\}} \left[-\phi m_{+1} + \beta V^b(m_{+1})\right] \tag{5}$$

Because of the envelope condition, $W^b(m)$ is linear in real balances, and therefore we can write $W^b(m) = \phi m + W^b(0)$.

Analogously, the value function of a seller, entering the CM with $\tilde{m}$, is defined by:

$$W^s(\tilde{m}) = \phi\tilde{m} + \max_{\{\tilde{m}_{+1}\}} \left[-\phi\tilde{m}_{+1} + \beta V^s(\tilde{m}_{+1})\right] \tag{6}$$

Each seller is allowed to post general contracts. In particular, we assume that the seller has the power to commit to an arbitrary extensive game tree, where the players are the seller and the buyer. The seller also finds advantageous to commit to a strategy and to announce such a strategy before the game begins. Obviously, we also require that the buyer finds in his interest to participate in the game. It is well known that, in this class of games, the seller's choice set is very large. Fortunately, however, we can apply the revelation principle to restrict, without loss of generality, our attention to an incentive-compatible and individually rational direct mechanism, where the buyer is asked to truthfully report his own type θ. Since the good is indivisible, a seller can try to discriminate among buyers by offering the good with different probabilities to different types. Conditional on a buyer's report, the mechanism therefore specifies the allocation probabilities $\alpha(\theta)$ and the real (i.e. expressed in terms of the CM good) money transfer $p(\theta)$, $\forall\theta \in [\underline{\theta}, \overline{\theta}]$, such that:

$$\begin{aligned} \alpha &: [\underline{\theta}, \overline{\theta}] \to [0,1] \\ p &: [\underline{\theta}, \overline{\theta}] \to \mathbb{R} \end{aligned}$$

This means that the seller commits to transferring the good to the buyer with probability $\alpha(\theta)$ if the buyer reports his type, and the buyer has to pay the seller a price $p(\theta)$.[6]

The value functions of buyers and sellers in the DM depend on the submarket they visit in equilibrium, and on their money holdings. For notational convenience, we ignore the dependence of the value function on the submarket. In equilibrium, the value of all active submarkets will be the same. A submarket is characterised by a buyers/sellers ratio and a pair of price and allocation probabilities.

Let $\mathbb{I}(\phi m \geq p(\theta))$ denote an indicator function which gives 1 if $\phi m \geq p(\theta)$ and 0

[6]It is well-known that the revelation principle is only applicable to settings in which the mechanism designer is able to credibly commit to any outcome of the mechanism (see Bester and Stausz [3]).

otherwise. The Bellman equation of a buyer can be written as:

$$
\begin{aligned}
V^b(m) &= \frac{\sigma(n)}{n}\int_{\underline{\theta}}^{\overline{\theta}} \left[\alpha(\theta)\theta + W^b(m - p(\theta)/\phi)\right] \mathbb{I}(\phi m \geq p(\theta)) dF(\theta) + \\
&\quad + \left[\frac{\sigma(n)}{n}\mathbb{I}(\phi m < p(\theta)) + 1 - \frac{\sigma(n)}{n}\right]\int_{\underline{\theta}}^{\overline{\theta}} W^b(m) dF(\theta) \\
&= \frac{\sigma(n)}{n}\int_{\underline{\theta}}^{\overline{\theta}} \left[(\alpha(\theta)\theta - p(\theta))\right] \mathbb{I}(\phi m \geq p(\theta)) dF(\theta) + \phi m + W^b(0) \qquad (7)
\end{aligned}
$$

The Bellman equation of a seller in the DM, who starts the period with cash balances $\tilde{m}$ is given by:

$$
\begin{aligned}
V^s(\tilde{m}) &= \sigma(n)\int_{\underline{\theta}}^{\overline{\theta}} -\alpha_{+1}(\theta)c + W^s(\tilde{m} + \tilde{m}_{+1}(\theta)) dF(\theta) + (1 - \sigma(n))W^s(\tilde{m}) \\
&= \sigma(n)\int_{\underline{\theta}}^{\overline{\theta}} \left[-\alpha_{+1}(\theta)c + p_{+1}(\theta)\right] dF(\theta) + W^s(\tilde{m}) \qquad (8)
\end{aligned}
$$

Substituting (7) evaluated at $+1$ into (5) we obtain:

$$
\begin{aligned}
&W^b(m) = \phi\,(m + T) + \max_{\{m_{+1}\}} -\phi m_{+1} + \beta \times \\
&\times \left[\frac{\sigma(n)}{n}\int_{\underline{\theta}}^{\overline{\theta}} \left[(\alpha_{+1}(\theta)\theta - p_{+1}(\theta))\right] \mathbb{I}(\phi m \geq p(\theta)) dF(\theta) + \phi_{+1} m_{+1} + W^b(0)\right] \qquad (9)
\end{aligned}
$$

A buyer will bring cash in the DM if $W^b(m_{+1})$, the continuation value from accumulating an amount of cash $m_{+1} \geq 0$, is greater than $W^b(0)$, the continuation value from accumulating an amount of cash $m_{+1} = 0$. Defining $\phi_{+1} m_{+1} = z_{+1}$ and using the Fisher equation $(1+i) = (1+r)\mu$ with $\beta = (1+r)^{-1}$, to derive the nominal interest rate i, condition (9) can be rewritten as:

$$
- i z_{+1} + \frac{\sigma(n)}{n}\int_{\underline{\theta}}^{\overline{\theta}} \left(\alpha_{+1}(\theta)\theta - p_{+1}(\theta)\right) dF(\theta) = \Omega \qquad (10)
$$

This equation states that sellers compete with each other to ensure each buyer a surplus of Ω. The value of Ω will be determined endogenously in equilibrium.

5 Optimal contracts

We now analyse the optimal choice of a seller with respect to the allocation probabilities and the price, focusing on steady-state equilibria, where $\alpha_{+1}(\theta) = \alpha(\theta)$, $p_{+1}(\theta) = p(\theta)$ and the gross rate of return on money is $\phi_{+1}/\phi = M/M_{+1} = \mu^{-1}$.

The competitive search market proceeds as follows. During the CM, while buyers can still rebalance their money holdings, sellers post their trading offers. Agents have rational expectations regarding the number of buyers who will be attracted by each offer and, hence, about the buyers/sellers ratio in each submarket. Since the realisation of θ is, for buyers, private information, the offers posted by sellers must be incentive compatible, i.e.:

$$\theta' \in \arg\max_{\theta \in [\underline{\theta}, \overline{\theta}]} \left[\alpha(\theta)\theta' - p(\theta)\right] \quad \forall \theta' \in [\underline{\theta}, \overline{\theta}] \tag{11}$$

Let now $v(\theta) = \alpha(\theta)\theta - p(\theta)$ denote the ex-post trading surplus for a type-θ buyer. We can then apply Proposition 23.D.2 Mas-Colell, Winston and Green [27] to state the following:

Lemma 1 *A trading offer satisfies the incentive compatibility constraint (11) iff: i) $\alpha(\theta)$ is non-decreasing and ii) $v(\theta)$ satisfies:*

$$v(\theta) - v(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} \alpha(x)dx \tag{12}$$

Lemma 1 tells us that the expected surplus of a type-θ buyer is pinned down by the function $\alpha(\theta)$ and by the expected surplus of the lowest type, $\underline{\theta}\alpha(\underline{\theta}) - p(\underline{\theta})$. It is easy to show that the participation constraint of the lowest type binds at the equilibrium.[7] Therefore, $\alpha(\underline{\theta})\underline{\theta} - p(\underline{\theta}) \leq \alpha(c)c - p(\underline{\theta}) \leq 0$ and participation for the $\underline{\theta}$-type requires

$$v(\underline{\theta}) \equiv \alpha(\underline{\theta})\underline{\theta} - p(\underline{\theta}) = 0 \tag{13}$$

Plugging this into (12) we get:

$$p(\theta) = \alpha(\theta)\theta - \int_{\underline{\theta}}^{\theta} \alpha(x)dx \tag{14}$$

Equation (14) shows that the price paid by a type θ buyer is given by the difference between his expected utility $\alpha(\theta)\theta$ and a term that reflects the surplus a seller must give the buyer in order to induce him to reveal his type.[8]

Keeping this in mind, we can write the seller's expected surplus as:

$$\Pi = \sigma(n) \int_{\underline{\theta}}^{\overline{\theta}} \left[-\alpha(\theta)c + \alpha(\theta)\theta - v(\theta)\right] dF(\theta) \tag{15}$$

Let now M denote the set of all submarkets that are active in equilibrium. An element $\mathsf{m} \in \mathsf{M}$ is a list $\{n_{\mathsf{m}}, \alpha_{\mathsf{m}}(\theta), p_{\mathsf{m}}(\theta)\}$. A competitive search equilibrium in the DM is a set $\{\mathit{M}, \Omega, \Pi\}$, such that, $\forall\ \mathit{m} \in \mathit{M}$: i) all buyers attain the same expected surplus Ω; ii) all sellers attain the same expected surplus Π; iii) the list m solves the

[7] We know in fact that in equilibrium $p(\theta) \geq \alpha(c)c$, $\forall\theta$. Since we assumed that $\underline{\theta} \leq c$ and $\alpha(\theta)$ is non-decreasing, we immediately see that $\alpha(\underline{\theta})\underline{\theta} \leq \alpha(c)c$.

[8] This is the famous *Revenue Equivalence Theorem* of auction theory (Myerson [30]) applied to our simple monopolistic screening problem.

following program:

$$\max_{\{\alpha(\theta),n\}} \quad \sigma(n)\int_{\underline{\theta}}^{\overline{\theta}}\left[-\alpha(\theta)c+\alpha(\theta)\theta-v(\theta)\right]dF(\theta) \tag{16}$$

$$\text{s.t.} \quad -iz+\frac{\sigma(n)}{n}\int_{\underline{\theta}}^{\overline{\theta}}v(\theta)dF(\theta)=\Omega \tag{17}$$

$$z\geq\alpha\left(\theta\right)\theta-v(\theta) \tag{18}$$

$$\dot{v}(\theta)=\alpha(\theta) \tag{19}$$

$$\alpha\left(\theta\right)\geq 0 \tag{20}$$

$$1\geq\alpha\left(\theta\right) \tag{21}$$

Conditions i) and ii) are straightforward. Since buyers are identical among each other, at the equilibrium they cannot attain different surpluses. The same goes for the sellers. Condition iii) states that sellers choose the optimal trading offer and the buyers/sellers ratio in the submarket in order to maximise their expected surplus Π, taking as given the buyers' expected surplus Ω, which endogenously adjusts to satisfy equation (17). The remaining four sets of constraints are the cash constraints (18), the incentive constraints (19), and the constraints on the allocation probabilities (20) and (21), respectively.

Problem (16)-(21) is an optimal control problem in which $v(\theta)$ is the state variable and $\alpha(\theta)$ is the control variable. We solve the problem in two steps. In the first, we take n, z and the Lagrange multiplier λ of constraint (17) as given and we solve for $\alpha\left(\theta\right)$ as a function of n, z, λ ignoring, for the moment, constraint (18). Then, we find the equilibrium values for n, z and λ using the solution of the first step and we verify that indeed this solution satisfies constraint (18).

Let now $\hat{\theta}$ solve:

$$\left(-c+\theta\right)f(\theta)+\left(\lambda\frac{\sigma(n)}{n}-1\right)\left(1-F(\theta)\right)=0 \tag{22}$$

We proceed by conjecturing that $\hat{\theta}$ exists and is unique. We will show below that this is true at the equilibrium. From the solution of the first step we obtain the following:

Lemma 2 *The optimal mechanism is non-stochastic:*

$$\hat{\alpha}=\begin{cases}1 & if \quad \theta\geq\hat{\theta}\\ 0 & if \quad \theta<\hat{\theta}\end{cases} \tag{23}$$

and implies a seller posting a price:

$$\hat{p}=\theta-\int_{\hat{\theta}}^{\theta}dx=\hat{\theta} \tag{24}$$

According to Lemma 2, a seller will post a price $\hat{p}$ and all the buyers that are willing to pay this price will be served. In equilibrium, a buyer that observes a posted price $\hat{p}$ will bring into the DM the amount of cash that just allows him to acquire the good. Therefore, we get that the cash constraint (18) is satisfied simply by setting:

$$z = \hat{p} = \hat{\theta} \tag{25}$$

This is a crucial result of our model. Because of indivisibility, the only way a seller can screen among different types of buyers is to use prices and allocation probabilities. Since the Lagrangian associated to problem (16)-(21) is linear in $\alpha(\theta)$, the seller's problem admits only corner solutions: $\alpha(\theta) = 1$ or $\alpha(\theta) = 0$. Therefore, the good is offered at a single price to all buyers, with probability 1. Below this price no trade occurs. Because of this, it will be optimal for buyers to bring just the cash necessary to acquire the DM good. Constraint (17) can be then rewritten as:

$$-i\hat{p} + \frac{\sigma(n)}{n} \int_{\hat{p}}^{\overline{\theta}} \big(\theta - \hat{p}\big) dF(\theta) = \Omega \tag{26}$$

By using equations (25) and (26), we can reduce the seller's problem (16)-(21) to:

$$\begin{aligned} \max_{\{n,\hat{p}\}} \quad & \sigma(n) \int_{\hat{p}}^{\overline{\theta}} (\hat{p} - c)\, dF(\theta) \\ \text{s.t.} \quad & (26) \end{aligned} \tag{27}$$

From the first order conditions we get:

$$\begin{aligned} & -\frac{1 - F(\hat{p})}{f(\hat{p})}(\hat{p} - c)\left[\frac{in}{\sigma(n)} + (1 - F(\hat{p}))\right] + \\ & + \left[\frac{1 - F(\hat{p})}{f(\hat{p})} - (\hat{p} - c)\right][\eta(n) - 1] \int_{\hat{p}}^{\overline{\theta}} (\theta - \hat{p}) dF(\theta) = 0 \end{aligned} \tag{28}$$

where $\eta(n) \equiv \frac{\sigma(n)}{\sigma'(n)n}$. To characterise the equilibrium we solve the system of equations (26)-(28) in two different cases: in the first the number of buyers is bounded while in the second is unbounded.

Before proceeding to the characterisation of the equilibrium, it is interesting to notice that the same results could be obtained by assuming that the good is divisibile, preferences are linear, i.e. $u(q(\theta)) = \theta q(\theta)$, and there is a capacity constraint, so that $q(\theta) \in [0, 1]$. Also in this case, the optimal contract implies that sellers optimally post a single price, below which there is no trade.

6 Limited number of buyers

6.1 Equilibrium

We now consider an equilibrium in which $n \leq N$, i.e. the measure of buyers who enter the DM is not too large, and may therefore coincide with the set of all potential buyers.[9] The optimal buyers/sellers ratio, n^*, must be consistent with the buyers' free entry condition:

$$-ip^* + \frac{\sigma(n^*)}{n^*}\int_{p^*}^{\overline{\theta}} \big(\theta - p^*\big)dF(\theta) \geq 0 \tag{29}$$

When, given a nominal rate i, all buyers enter the DM, $n^* = N$ and equation (29) is satisfied as a strict inequality. In this case,

$$-ip^* + \frac{\sigma(N)}{N}\int_{p^*}^{\overline{\theta}} \big(\theta - p^*\big)dF(\theta) = \Omega^* \tag{30}$$

determines the equilibrium value of the buyers' expected surplus Ω^*. When instead the number of buyers who enter the DM is lower than the economy-wide N, the free entry condition (29) holds as an equality, and $p^*(i)$ and $n^*(i)$ are jointly determined by (28) and (29) holding as an equality.

Looking at equation (28), we can immediately see that it always admits a solution for $p^* = \overline{\theta}$. As in most models of money, also in this economy, autarky is a possible equilibrium. Keeping this in mind, we concentrate on the more interesting monetary equilibrium, which we define as follows:

Definition 1 *Given a monetary policy i, a symmetric, stationary monetary equilibrium (SSME) is a CM allocation (X, H) and a DM outcome (p^*, n^*, Ω^*) such that:*

1. *(X, H) solves (5) and (6) with $m_{+1} = 0$ for sellers, $m_{+1} = M_{+1}$ for buyers and $X = H$;*
2. *p^*, n^* and Ω^* solve (28) and (30), with $n^* = N$ if (29) holds as a strict inequality and $\Omega^* = 0$ otherwise;*
3. *$z = \phi M$ satisfies the market clearing condition $m = M = p^*/\phi$.*

In order to characterise the equilibrium, we first show that there exists a level of i, that we denote i^N, below which all potential buyers enter the market and $n^* = N$. There also exists a maximum level of the nominal rate, i^C, above which there is no trade in the DM.

Lemma 3 *There exist i^N and i^C such that: (i) for $i \in [0, i^N)$, $n^* = N$; (ii) for $i \in [i^N, i^C)$, $n^* < N$, with n^* defined by (29) holding as an equality, and (iii) for $i \in [i^C, \infty]$, $n^* = 0$.*

We can now state the following:

[9] In this case, we assume that money is injected through lump-sum transfers to all potential buyers.

Proposition 1 *A SSME always exists for* $i \in [0, i^C)$*.*

Proposition 2 *For* $i \in [0, i^N)$*, the SSME is unique and* $dp^*/di < 0$*.*

Given Lemma 2, at the equilibrium $p^* = \hat{\theta}$, hence the conjecture we made above on the uniqueness of $\hat{\theta}$ is verified. Proposition 2 represents an important result of our model: when $i \in [0, i^N)$, all buyers enter the DM and the buyers/sellers ratio is N. In this case, there always exists a unique monetary equilibrium at which an increase in the rate of inflation (that given the Fisher equation translates into an increase in the nominal rate of interest) causes a reduction in the price of the indivisible good. When $i \in [i^N, i^C)$, it is difficult to draw general conclusions on the effect of i on p^* and n^*. However, if we restrict our attention to constant elasticity matching functions, we can easily prove the following:[10]

Proposition 3 *When* $\sigma(n)$ *is isoelastic and* $i \in [i^N, i^C)$*, the SSME is unique,* $dp^*/di = 0$ *while* $dn^*/di < 0$*.*

Given Lemma 2, also Proposition 3 verifies the conjecture on the uniqueness of $\hat{\theta}$. This Proposition simply states that above i^N, a higher inflation will negatively affect the entry of buyers into the DM.

Propositions 2 and 3 tell us that, for $i < i^N$, all buyers participate in the DM and the equilibrium price is decreasing in the nominal rate while for $i \geq i^N$, the number of buyers decreases as the nominal rate increases and the equilibrium price does not change (for the case of a constant elasticity matching function). To understand this, let us start from i close to zero; in this case, since the cost of entering the market is very low, all buyers participate and, in order to attract costumers in their submarket, sellers compete by offering buyers a positive surplus. When i goes up, the cost of holding money increases and, to keep buyers entering their submarkets, sellers find it optimal to reduce p^*. However, the reduction in p^* does not fully compensate the increase in i, and so, at the same time, Ω^* also declines. The nominal rate will eventually reach a point at which the buyers' expected surplus becomes zero. Beyond this point, some buyers will be discouraged from entering the DM and the buyers/seller ratio declines. Since now $n^* < N$, sellers do no longer lower the price of the good and the burden of adjustment falls entirely on the buyers/sellers ratio. Eventually, if i keeps increasing, it will reach a level i^C, at which the market shuts down.

These results have interesting implications for aggregate economic activity. The total quantity of indivisible goods traded in the DM is given by

$$Q = \sigma\left(n^*\right) \int_{p^*}^{\overline{\theta}} dF(\theta) \tag{31}$$

[10] An example of a matching function which is suitable for our purpose is $\sigma(n) = An^{\frac{1}{\eta}}$ with $\eta < 1$ and $A \leq \max\{N^{-\frac{1}{\eta}}, N^{\frac{\eta-1}{\eta}}\}$. This last condition ensures that both n^{α} and $n^{\alpha-1}$ do not exeed 1.

When $i < i^N$, monetary policy does not have any impact on the buyers/sellers ratio and $n^* = N$ but affects the number of trades which effectively occur in the DM, by influencing the level of p^*. Differentiating totally (31) we get

$$\frac{dQ}{di} = -\sigma(N)\frac{dp^*}{di}f(p^*) > 0 \tag{32}$$

Again, the logic behind this result is quite straightforward. After θ is revealed, a buyer decides whether to acquire the DM good or not by simply comparing the utility he derives from consumption and the price of the good. Since a higher i implies a lower price, more buyers are willing to acquire the good and aggregate output increases.

When instead, $i \in [i^N, i^C)$, the optimal buyers/sellers ratio is determined by (29) holding as an equality. Unfortunately, for this range of the nominal rate, we cannot unambiguously determine the effect of i on aggregate output. An interesting case, analysed in Proposition 3, is when the matching function is isoelastic. In this case, monetary policy affects the buyers/seller ratio while p^* does no longer depend on i. Differentiating totally (31) we get:

$$\frac{dQ}{di} = \sigma'(n^*)\frac{dn^*}{di}(1 - F(p^*)) < 0 \tag{33}$$

where $dn^*/di = -p^* \left[\frac{\sigma(n)}{n}\left(\frac{\eta-1}{\eta}\right)\int_{p^*}^{\overline{\theta}}(\theta - p^*)dF(\theta)\right]^{-1} < 0$. This result simply reflects the effect that an increase in i has on buyers' entry decision. When i goes up, fewer buyers show up in the DM. Sellers produce less and aggregate output goes down.

Example In order to obtain a closed form solution for the equilibrium price p^*, suppose that θ is distributed uniformly with density $f(\theta) = \frac{1}{\overline{\theta}-\underline{\theta}}$ and $\sigma(n)$ is isoelastic. In this case, conditions (28) and (30) can be combined as follows

$$-(\overline{\theta} - \hat{p})(\hat{p} - c)\left[i\frac{N}{\sigma(N)} + \frac{\overline{\theta} - \hat{p}}{\Delta\theta}\right] + \left(\overline{\theta} - 2\hat{p} + c\right)\frac{(\overline{\theta} - \hat{p})^2}{2\Delta\theta} = 0 \tag{34}$$

Let now q indicate $(\overline{\theta} - \hat{p})$. Excluding the solution $p^* = \overline{\theta}$, which implies no trade in the DM, equation (34) becomes:

$$-q[(\overline{\theta} - c) - q]\left(i\frac{N}{\sigma(N)} + \frac{q}{\Delta\theta}\right) + [2q - (\overline{\theta} - c)]\frac{q^2}{2\Delta\theta} = 0$$

which can be rewritten as:

$$2q^2 - \left(\frac{3}{2}(\overline{\theta} - c) - i\frac{\Delta\theta N}{\sigma(N)}\right)q - (\overline{\theta} - c)i\frac{\Delta\theta N}{\sigma(N)} = 0 \tag{35}$$

which as a unique positive root:

$$p^* = \overline{\theta} - \frac{1}{4}\left(\frac{3}{2}(\overline{\theta} - c) - i\frac{\Delta\theta N}{\sigma(N)}\right) - \frac{1}{4}\left[\left(\frac{3}{2}(\overline{\theta} - c) - i\frac{\Delta\theta N}{\sigma(N)}\right)^2 + 8(\overline{\theta} - c)i\frac{\Delta\theta N}{\sigma(N)}\right]^{1/2} \tag{36}$$

Notice that, for $n = N$, $\nexists\, i \in [0,\infty)$ s.t. $p^* = c$. After few algebraic steps, we can easily check that, when $p^* < \overline{\theta}$, the function $p^*(i)$ is decreasing in i and is equal to $(\overline{\theta} + 3c)/4$ when $i = 0$.

6.2 Optimal monetary policy

Usually, in micro-founded models of money, optimal monetary policy is found at the Friedman rule; we prove now that, in our economy, this is not necessarily true. To this purpose it is sufficient to consider the case in which the matching function is isoelastic.

We analyse the problem of a monetary authority that weighs all agents equally, and chooses the nominal rate of interest (or the rate of inflation) to maximise the sum of the expected utilities of buyers and sellers, subject to the same constraints the environment imposes on private agents. Hence, we define the welfare function as:

$$\mathcal{W} = \overline{\mathcal{B}}V^b + V^s$$

In equilibrium, the amount of cash a buyer brings into the DM is $z = p^*$. Using condition (30), a buyer's *ex ante* welfare V^b can be expressed as:[11]

$$\begin{aligned} V^b &= \frac{1}{1-\beta}\left(\Omega^* + ip^*\right) \\ &= \frac{1}{1-\beta}\frac{\sigma(n^*)}{n^*}\int_{p^*}^{\overline{\theta}} \left(\theta - p^*\right) dF(\theta) \end{aligned} \tag{37}$$

Analogously, a seller's *ex ante* welfare V^s is given by:[12]

$$V^s = \frac{1}{1-\beta}\left[\sigma(n^*)\left(-c + p^*\right)\left(1 - F(p^*)\right)\right] \tag{38}$$

[11] Combine equation (7) and Lemma 2 to get:

$$\begin{aligned} V^b(m) &= \frac{\sigma(n^*)}{n^*}\int_{p^*}^{\overline{\theta}} \left(\theta + W^b(m - p^*)\right) dF(\theta) + \frac{\sigma(n^*)}{n^*}\int_{\underline{\theta}}^{p^*} W^b(m) dF(\theta) + \left(1 - \frac{\sigma(n^*)}{n^*}\right) W^b(m) \\ &= \frac{\sigma(n^*)}{n^*}\int_{p^*}^{\overline{\theta}} \left(\theta - p^*\right) dF(\theta) + W^b(m) \\ &= \frac{\sigma(n^*)}{n^*}\int_{p^*}^{\overline{\theta}} \left(\theta - p^*\right) dF(\theta) + (\phi m + \phi T) - (\phi m_{+1}) + \beta V^b(m_{+1}) \end{aligned}$$

Simplifying, at the steady state, we get (37).

[12] Combine equation (8) and Lemma 2 to get:

$$\begin{aligned} V^s(0) &= \sigma(n^*)\int_{p^*}^{\overline{\theta}} \left(-c + W^s(m)\right) dF(\theta) + \sigma(n^*)\int_{\underline{\theta}}^{p^*} W^s(0) dF(\theta) + \left(1 - \sigma(N)\right) W^s(0) \\ &= \sigma(n^*)\left(-c + p^*\right)\left(1 - F(p^*)\right) + W^s(0)\left\{\sigma(n^*)\left[1 - F(p^*) + F(p^*)\right] + \left(1 - \sigma(n^*)\right)\right\} \\ &= \sigma(n^*)\left(-c + p^*\right)\left(1 - F(p^*)\right) + W^s(0) \end{aligned}$$

Substituting $W^s(0) = \beta V^s(0)$ we get (38).

The monetary authority chooses the nominal rate of interest to maximise:

$$\mathcal{W} = \begin{cases} \frac{1}{1-\beta}\sigma(N)\int_{p^*}^{\overline{\theta}}(\theta - c)dF(\theta) & \text{if} \quad i \in [0, i^N) \\ \\ \frac{1}{1-\beta}\sigma(n^*)(p^* - c)(1 - F(p^*)) & \text{if} \quad i \in [i^N, i^C) \end{cases} \tag{39}$$

since when $i \in [i^N, i^C)$ the buyers' surplus goes to zero. The first order condition for an interior solution implies:

$$\frac{d\mathcal{W}}{di} = \begin{cases} -\frac{1}{1-\beta}\sigma(N)(p^* - c)f(p^*)\frac{dp^*}{di} > 0 & \text{if} \quad i \in [0, i^N) \\ \\ \frac{1}{1-\beta}\sigma(n^*)(p^* - c)(1 - F(p^*))\frac{dn^*}{di} < 0 & \text{if} \quad i \in [i^N, i^C) \end{cases} \tag{40}$$

Hence, we can state the following:

Proposition 4 *The optimal nominal interest rate is $i^N > 0$.*

Welfare is increasing in i up to i^N while it is decreasing in i for higher values of the nominal rate. When $i \in [0, i^N)$, the interest rate is so low that all buyers participate in the DM. In this range, an increase in i does not affect participation; moreover, the cost of holding money is rebated back to buyers through lump-sum transfers. The nominal rate, therefore, affects aggregate surplus only by affecting p^*. A higher i induces sellers to lower the price and this increases the number of agents who are willing to acquire the good, therefore increasing aggregate welfare. On the contrary, when $i \in [i^N, i^C)$, in this economy with isoelastic matching function, p^* does not depend on i; buyers' entry keeps their surplus to zero and an increase in inflation negatively affects sellers' surplus by reducing the number of trades.

7 Free entry of buyers

In this section, we consider the case of buyers' free entry, in which the measure of potential buyers $\mathcal{B}$ is unbounded but buyers face an entry cost k. For simplicity, we assume again that $\sigma(n)$ exhibits constant returns to scale.[13] We also assume that money is increased by the monetary authority through lump-sum transfers to sellers rather than to buyers.[14] In this case, at the steady-state equilibrium, the CM value function of a seller becomes:

$$W^s(0) = \phi T + \beta V^s(\tilde{m}_{+1}) \tag{41}$$

[13]In particular, assume $\sigma(n) = An^{\frac{1}{\eta}}$. In order to ensure that both $\sigma(n)$ and $\sigma(n)/n$ are less than 1, we restrict the analysis to values of A and i such that $A \leq \max\left\{\overline{n}^{-\frac{1}{\eta}}, \overline{n}^{\frac{\eta-1}{\eta}}\right\}$, where $\overline{n}$ is the equilibrium level of n defined below.

[14]This is done to circumvent the problem of unlimited lump-sum transfers to buyers.

where $\phi T = (\mu - 1)\hat{p}$. The equilibrium values for $\hat{p}$ and n are given by:

$$\begin{aligned} - & \left(\frac{1-F(\hat{p})}{f(\hat{p})}\right)(\hat{p}-c)\left[\frac{in}{\sigma(n)} + (1-F(\hat{p}))\right] + \\ + & \left[\left(\frac{1-F(\hat{p})}{f(\hat{p})}\right) - (\hat{p}-c)\right](\eta - 1)\int_{\hat{p}}^{\overline{\theta}}(\theta - \hat{p})dF(\theta) = 0 \end{aligned} \tag{42}$$

and

$$-i\hat{p} + \frac{\sigma(n)}{n}\int_{\hat{p}}^{\overline{\theta}}(\theta - \hat{p})dF(\theta) = k \tag{43}$$

Then, we define the equilibrium as:

Definition 2 *Given a monetary policy i, a SSME with an unlimited number of buyers is a CM allocation (X, H) and a DM outcome $(\overline{p}, \overline{n})$ such that:*

1. *(X, H) solves (5) with $T = 0$ and (41) with $m_{+1} = 0$ for sellers, $m_{+1} = M_{+1}$ for buyers and $X = H$;*
2. *$\overline{p}$, $\overline{n}$ solve (42) and (43);*
3. *$z = \phi M$ satisfies the market clearing condition $m = M = \overline{p}/\phi$.*

Defining $i^U = 1/c\left\{\sigma[n(c)]/n(c)\int_c^{\overline{\theta}}(\theta - c)dF(\theta) - k\right\}$, we can now state:

Proposition 5 *If $i \in [0, i^U]$, there exists a unique SSME with unlimited number of potential buyers, and $d\overline{p}/di < 0$.*

which again verifies the conjecture on the uniqueness of $\hat{\theta}$. Differently from the previous case, buyers' free entry implies that, at the equilibrium, $d\overline{p}/di$ is always negative as long as $\overline{p} > c$. The reason for this is similar to the one analysed in the previous section. When i goes up, buyers' expected surplus goes down; in order to guarantee a surplus k to their costumers, sellers lower the price. The interest rate has an upper bound i^U at which $\overline{p} = c$. Notice that the nominal rate influences the equilibrium level of the buyers/sellers ratio both directly and indirectly, through its effect on the equilibrium price.

7.1 Optimal monetary policy

As in the previous section, we analyse the optimal intervention of a monetary authority that takes as given the equilibrium values $\overline{p}(i)$ and $\overline{n}(i)$, jointly determined by equations (42)-(43). The transfer given to the sellers $(\mu - 1)\overline{p}$ is paid by the buyers through the inflation tax $i\overline{pn}$ and, therefore, cancels out from aggregate welfare. The monetary authority chooses the nominal rate to maximise:

$$\mathcal{W} = \frac{1}{1-\beta}\left[\sigma(\overline{n})\int_{\overline{p}(i)}^{\overline{\theta}}(\overline{p} - c)\,dF(\theta) + \overline{n}\left(\frac{\sigma(\overline{n})}{\overline{n}}\int_{\overline{p}(i)}^{\overline{\theta}}(\theta - \overline{p})\,dF(\theta) - k\right)\right] \tag{44}$$

where $\overline{n}$ is implicitly defined by (43).

Before solving this problem and recalling that η is the inverse of the elasticity of the matching function, we prove the following:

Lemma 4 *When $i = 0$, buyers' free entry implies* $\frac{\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-\overline{p}(0))dF(\theta)}{\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-c)dF(\theta)} > \frac{1}{\eta}$.

Lemma 4 delivers an important result: because of asymmetric information, in this model, at the Friedman rule, the Hosios [21] condition is not satisfied, i.e. the share of surplus that goes to the entrants is greater than the elasticity of the matching function.[15] In light of this, we can state the following:

Proposition 6 *With buyers' free entry,*

$$sign\,(d\mathcal{W}/di)\,|_{i=0} = sign\left(\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-\overline{p}(0))dF(\theta) - \frac{1}{\eta}\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-c)dF(\theta)\right) > 0,$$

i.e. the Friedman rule is not optimal.

The explanation for the non-optimality of the Friedman rule in this case is quite different from the one we obtained in the previous section. If information were symmetric and $i = 0$, directed search would guarantee free entry of buyers and the attainment of the Hosios condition. With asymmetric information and $i = 0$, the buyers' share of surplus is too high, and too many buyers enter the DM. In this case, since $\partial\overline{n}/\partial i < 0$, an inflationary monetary policy causes the ratio of buyers to sellers to decline toward the efficient level.

8 Discussion

In order to better understand our results, consider that in this model buyers make two basic decisions. The first one is whether to acquire money and enter the DM, given the probability of matching with a seller. This decision is made by buyers before they know their type. The second one is whether to trade, once the match has occurred and the type revealed.

When the number of buyers is bounded, the price of the good is decreasing in i, up to i^N. This is true independently of asymmetric information, and is due to the competitive search environment we analyse in this paper. Consider, for example, the particular case studied by Han, Julien, Petursdottir and Wang [19], which also analyse a Rocheteau-Wright economy with competitive search and indivisible goods, but assume a deterministic θ observed by sellers; also in this case, $dp^*/di < 0$. The reason is straightforward: an increase in i increases the cost of holding real balances so that fewer buyers enter the DM; in order to induce participation, sellers lower the price. In the model of Han et al., however, changes in i in the interval $i \in [0, i^N]$

[15] As we show in the Proof of Lemma 4, this result does not depend on the assumption that the matching function is isoelastic.

do not have any consequence on welfare. As long as the economy is at a corner, i.e. $n = N$, participation is maximum and trade inside a match always occurs as long as $\theta \geq p^*$. Changes in i only affect the distribution of the surplus between buyers and sellers and optimal monetary policy, therefore, is obtained at any $i \in [0, i^N]$. In our model, instead, the optimal monetary policy is found at $i^N > 0$. As in the model with deterministic θ, an increase in i induces sellers to decrease p^*. This is a consequence of competition between sellers: to prevent buyers from moving away from their submarkets, sellers try to compensate the loss in surplus caused by the higher inflation with a lower price. Unlike the model with deterministic θ, however, this also affects the number of trades that occur when buyers and sellers are matched. A decrease in p^* increases the number of buyers who find it optimal to trade, and because of this effect on the extensive margin, the optimal monetary policy implies a positive nominal rate.

In our environment, because of asymmetric information and indivisibility, a seller, once he is matched, faces a downward sloping demand curve, as in the text-book model of the non-discriminating monopolist. Monetary policy can affect welfare by exploiting the negative relationship between price and aggregate quantity. To highlight the effect of asymmetric information, it is useful to compare our model with one in which agents' types are stochastic but perfectly observed by sellers and $i = 0$. It can be easily verified that, in this environment, all buyers would bring cash in the DM and sellers could extract the entire match surplus by charging, for any possible realisation of θ, a price which is equal to the buyer's evaluation of the good. Under symmetric information, therefore, the Friedman rule would maximise buyers' entry and, at the same time, perfect price discrimination would achieve efficiency.

When the number of buyers is unbounded, market tightness is endogenously determined by the buyers' free entry condition. In a competitive search environment with symmetric information, at the Friedman rule, the buyers/sellers ratio satisfies the Hosios condition, since sellers internalise the effect of n on matching probabilities. Under asymmetric information, instead, at $i = 0$, buyers' free entry causes inefficient $\overline{n}$ and the share of surplus that goes to the entrants is greater than the elasticity of the matching function. An inefficiently high number of buyers enter the DM, attracted by the informational rent they can enjoy when matched with sellers. Since at the optimum $\partial\overline{n}/\partial i < 0$, the monetary authority can discourage participation and contrast the congestion effect by increasing the cost of holding cash. The same effect is also found in one of the three full information models analysed by Rocheteau and Wright [31], the one with price-taking agents and sellers' free entry. Under competitive search, differently from us, Rocheteau and Wright find that deflation is optimal. Moreover, it is important to stress that in our model inflation affects welfare not only by contrasting eccessive entry, but also when the buyers/sellers ratio is exogenously given. In this case, the mechanism is different and works by reducing the monopoly power of sellers.

One of the key insights of our model is that markups decrease in response to an inflationary monetary policy, for not too high rates of inflation. In order to see whether this effect can be found in the data, we plot in Figure 2 the series of CPI inflation rate and aggregate markups (estimated by De Loecker and Eeckhout [9]), for the US in the period 1961-2014.[16] The graph clearly shows that the two series move in opposite directions, with a correlation coefficient of -0.738. This provide a vey rough preliminary evidence on the plausibility of the mechanism we highlight with our model.

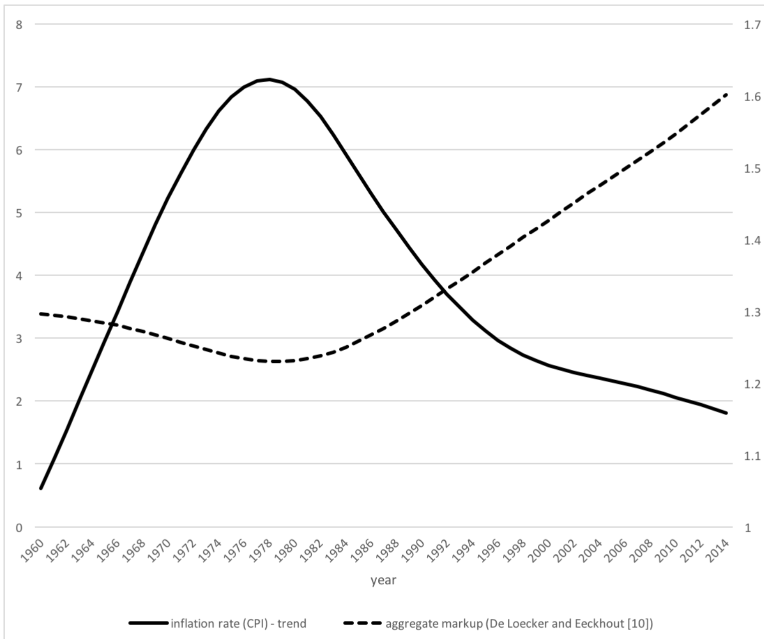

Figure 2: Markups and inflation in the long-run (US data)

9 Concluding remarks

This paper revisits some long standing issues in macroeconomics such as the existence of a stable output inflation trade-off and the optimality of the Friedman rule. Independently of whether a negative correlation between inflation and aggregate output can be found in the data - a yet not settled issue we do not address in this paper - we ask whether in a fully micro-founded model of money without assuming nominal rigidities, fully anticipated inflation may have positive effects on aggregate output and central bank may find optimal to set a nominal interest rate above zero.

Our answer is yes. Building on Rocheteau and Wright [31], we develop a competitive search model with indivisible goods in which sellers post optimal contracts. We show that the existence of asymmetric information is sufficient to justify the need for an active monetary policy. With a bounded number of buyers inflation lowers the price of the indivisible good and monetary policy can exploit this effect, increasing aggregate output and welfare. When the number of buyers is unbounded an

[16]The trend components of these series were extracted using the Hodrick-Prescott filter, with a smoothing parameter of 677.13.

active monetary policy is needed to correct the congestion effect created by buyers' excessive entry.

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Appendix

Proof of Lemma 1 A direct mechanism is i) incentive compatible iff $\alpha(\theta)\theta - p(\theta) \geq \alpha(\theta')\theta - p(\theta')$, $\forall \theta, \theta' \in [\underline{\theta}, \overline{\theta}]$, and is ii) individually rational iff $\alpha(\theta)\theta - p(\theta) \geq 0$, $\forall \theta \in [\underline{\theta}, \overline{\theta}]$. Consider two types θ', $\theta \in [\underline{\theta}, \overline{\theta}]$, with $\theta > \theta'$. Incentive compatibility requires:

$$\begin{aligned} \alpha(\theta)\theta - p(\theta) &\geq \alpha(\theta')\theta - p(\theta') \\ \alpha(\theta)\theta' - p(\theta) &\leq \alpha(\theta')\theta' - p(\theta') \end{aligned}$$

Adding up the two conditions we get:

$$\left(\theta - \theta'\right)\left(\alpha(\theta) - \alpha(\theta')\right) \geq 0$$

which implies that $\alpha(\theta)$ has to be non-decreasing. For simplicity, here, we restrict our attention to differentiable function.

In turn, incentive compatibility implies that the optimal response, $\tilde{\theta}$, chosen by a type-θ buyer satisfies:

$$\dot{\alpha}(\tilde{\theta})\theta - \dot{p}(\tilde{\theta}) = 0$$

For the truth to be an optimal response for all θ, we must have:

$$\dot{\alpha}(\theta)\theta - \dot{p}(\theta) = 0 \tag{45}$$

which must hold for all $\theta \in [\underline{\theta}, \overline{\theta}]$ since θ is unknown to the seller. The second order condition requires that $\dot{\alpha}(\theta) \geq 0$. Recalling that $v(\theta) = \alpha(\theta)\theta - p(\theta)$, the incentive constraint (45) implies that:

$$\dot{v}(\theta) = \alpha(\theta) \tag{46}$$

Integrating both sides of (46) between θ and $\underline{\theta}$ we get:

$$v(\theta) - v(\underline{\theta}) = \int_{\underline{\theta}}^{\theta} \alpha(x)dx$$

which is equation (12).

Proof of Lemma 2 The Hamiltonian for the optimal control problem (16)-(21) is then given by:

$$\mathcal{H} = \left[-\alpha(\theta)c + \alpha(\theta)\theta - v(\theta)\right] f(\theta) + \mu(\theta)\alpha(\theta) \tag{47}$$

where $\mu(\theta)$ is the co-state variable. The Lagrangian is:

$$\begin{aligned} \mathcal{L} \;=\;& \left[-\alpha(\theta)c + \alpha(\theta)\theta - v(\theta)\right] f(\theta) + \lambda\left(\frac{\sigma(n)}{n} v(\theta) f(\theta)\right) + \mu(\theta)\alpha(\theta) + \\ &+ \nu_1(\theta)\alpha(\theta) + \nu_2(\theta)\left(1 - \alpha(\theta)\right) \end{aligned} \tag{48}$$

The first order conditions, $\forall\theta \in [\underline{\theta}, \overline{\theta}]$ are:

$$\alpha(\theta): \qquad (-c+\theta)\, f(\theta) + \mu(\theta) + \nu_1(\theta) - \nu_2(\theta) = 0 \tag{49}$$

$$v(\theta): \qquad \left(\lambda \frac{\sigma(n)}{n} - 1\right) f(\theta) = -\dot{\mu}(\theta) \tag{50}$$

The complementary slackness conditions are given by

$$\mu(\theta)\alpha(\theta) = 0 \tag{51}$$

$$\lambda \left(\frac{\sigma(n)}{n} v(\theta) f(\theta)\right) = 0 \tag{52}$$

$$\nu_1(\theta)\alpha(\theta) = 0 \tag{53}$$

$$\nu_2(\theta)\left(1 - \alpha(\theta)\right) = 0 \tag{54}$$

The transversality condition requires that $\mu(\overline{\theta})v(\overline{\theta}) = 0$, which in turn implies $\mu(\overline{\theta}) = 0$. Integrating both sides of (50) between θ and $\overline{\theta}$ yields:

$$\mu(\theta) = \lambda \frac{\sigma(n)}{n}(1 - F(\theta)) \tag{55}$$

Using (55), we can rewrite (49) as:

$$(-c+\theta)\, f(\theta) + \left(\lambda \frac{\sigma(n)}{n} - 1\right)(1 - F(\theta)) + \nu_1(\theta) - \nu_2(\theta) = 0 \tag{56}$$

When $\nu_2(\theta) > 0$, $\nu_1(\theta) = 0$ and

$$(-c+\theta)\, f(\theta) + \left(\lambda \frac{\sigma(n)}{n} - 1\right)(1 - F(\theta)) > 0 \iff \alpha(\theta) = 1 \tag{57}$$

On the other side, when $\nu_2(\theta) = 0$, $\alpha(\theta) = 0$, $\nu_1(\theta) > 0$ and

$$(-c+\theta)\, f(\theta) + \left(\lambda \frac{\sigma(n)}{n} - 1\right)(1 - F(\theta)) < 0 \iff \alpha(\theta) = 0 \tag{58}$$

Since $\alpha(\theta)$ can only be 0 or 1, we immediately see that it is never optimal for the seller to use allocation probabilities to discriminate among buyers.

Using equation (12), we obtain that $\hat{p} = \hat{\theta}$.

Proof of Lemma 3 Define $H(N,i) \equiv -ip^*(i) + \sigma(N)/N \int_{p^*(i)}^{\overline{\theta}} (\theta - p^*(i))dF(\theta)$ where p^* solves (28) for $n = N$. Notice that $H(N,i)$ is continuous in i and $H(N,0) = \sigma(N)/N \int_{p^*(0)}^{\overline{\theta}} (\theta - p^*(0))dF(\theta) \geq 0$ while $H(N,\infty) < 0$. Hence, by virtue of the intermediate value theorem, there must exist a $i^N \in [0,\infty)$ such that $H(N,i) = 0$. To prove that there is a unique i^N below which $H(N,i) < 0$, we need to show that H is a non-increasing function of i. To do this we use an argument similar to that contained in Mattesini and Nosal [29]. Suppose that for $i = i_1$, $H(N,i_1)$ is a local maximum and suppose there is another level of i, $i_2 > i_1$, such that

$H(N, i_1) = H(N, i_2)$. As i increases, say to $i_1 + \epsilon$, a buyer will not find it convenient to increases his money balances, since $H(N, i_1 + \epsilon) < H(N, i_1) = H(N, i_2)$. He will keep then the same level of cash as long as $i \leq i_2$, while he will find it convenient to change it for $i > i_2$. This implies that $dH/di \leq 0$ and there exists a unique i^N such that $H(N, i^N) = 0$.

To prove that there exists an upper bound on the nominal rate, i^C, such that $n^* = 0$, notice that when $n^* \to 0$, $\sigma'(n) \to 1$. Applying Hôpital rule, we can easily see that, for $n^* = 0$ the first order condition (28) becomes: $-\frac{(1-F(\hat{p}))}{f(\hat{p})}(\hat{p} - c)\left[i + (1 - F(\hat{p}))\right] = 0$, which is satisfied only when $p^* = c$ or $p^* = \overline{\theta}$. We know that $p^* = \overline{\theta}$ is not a monetary equilibrium, hence, substituting $p^* = c$ into equation (29) we get: $i^C = 1/c \int_c^{\overline{\theta}} (\theta - c) dF(\theta)$. For $i \geq i^C$, $n^* = 0$.

Proof of Proposition 1 Denote with $\Psi(\hat{p})$ the LHS of equation (28). We want to prove the existence of a $\hat{p} = p^* \in [c, \overline{\theta})$ that solves $\Psi(\hat{p}) = 0$.

We first show that there always exists a $\tilde{p}$ such that

$$\xi(\tilde{p}) \equiv \left(\frac{1 - F(\tilde{p})}{f(\tilde{p})} - (\tilde{p} - c) \right) = 0 \tag{59}$$

This can be easily seen by observing that $\xi(c) = (1 - F(c))/c > 0$, $\xi(\overline{\theta}) = -(\overline{\theta} - c) < 0$ and $\xi'(\tilde{p}) = \frac{d}{d\tilde{p}} \left(\frac{1-F(\tilde{p})}{f(\tilde{p})} - (\tilde{p} - c) \right) < 0$, which implies the function $\xi(\tilde{p})$ must cross the horizontal axis only once, so that there is a unique solution for (59).

We first consider the value of Ψ for $\hat{p} = \tilde{p}$ both when $i \in [0, i^N)$ and $i \in [i^N, i^C)$. We see that when $i \in [0, i^N)$, $n = N$ and

$$\Psi(\tilde{p}) = -\frac{1 - F(\tilde{p})}{f(\tilde{p})} (\tilde{p} - c) \left[\frac{iN}{\sigma(N)} + (1 - F(\tilde{p})) \right] < 0 \tag{60}$$

When instead, $i \in [i^N, i^C)$, $n = n^* < N$ and condition (29) holds as an equality so that:

$$\Psi(\tilde{p}) = -\frac{1 - F(\tilde{p})}{f(\tilde{p})} (\tilde{p} - c) \left[\frac{\int_{\tilde{p}}^{\overline{\theta}} (\theta - \tilde{p}) dF(\theta)}{\tilde{p}} + (1 - F(\tilde{p})) \right] < 0 \tag{61}$$

Moreover, for $n \leq N$:

$$\Psi(c) \equiv \frac{1 - F(c)}{f(c)} (\eta - 1) \int_c^{\overline{\theta}} (\theta - c) dF(\theta) > 0 \tag{62}$$

This implies that, for $i \in [0, i^C)$, i.e. $n \leq N$, the function $\Psi(\hat{p})$ starts from a positive value, approaches 0 from above, and then assumes negative values; therefore we have

$$\left.\frac{d\hat{p}}{di}\right|_{\hat{p}=p^*} = -\frac{-\frac{1-F(p^*)}{f(p^*)}(p^* - c)\frac{N}{\sigma(N)}}{\Psi'(p^*)} < 0 \tag{63}$$

Since the function is continuous, by virtue of the intermediate value theorem, we can then conclude that there exists a $p^* \in (c, \overline{\theta})$ that solves $\Psi(\hat{p}) = 0$.

Proof of Proposition 2 In the proof of Proposition 1 we showed that, as long as $\hat{p}$ goes from c to $\tilde{p}$, the function Ψ moves from positive to negative values, i.e. there exists at least a $\hat{p} = p^*$ such that $\Psi(p^*) = 0$ and $\Psi'(p^*) < 0$. We now derive the conditions under which p^* is also unique. Evaluating again (28) at $\hat{p} = p^*$ and differentiating with respect to $\hat{p}$ we get:

$$\begin{aligned}
- & \left[\left.\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}}\right|_{\hat{p}=p^*} (p^*-c) + \left(\frac{1-F(p^*)}{f(p^*)}\right)\right] \left[\frac{iN}{\sigma(N)} + (1-F(p^*))\right] + \\
+ & \left[-\left(\frac{1-F(p^*)}{f(p^*)}\right)(p^*-c)\right](-f(p^*)) + \\
+ & \left[\left.\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}}\right|_{\hat{p}=p^*} - 1\right](\eta-1)\int_{p^*}^{\overline{\theta}}(\theta-p^*)dF(\theta) + \\
- & \left[\left(\frac{1-F(p^*)}{f(p^*)}\right) - (p^*-c)\right](1-F(p^*))
\end{aligned} \tag{64}$$

which is the slope of Ψ when it crosses the horizontal axis. We show now that this is always negative, i.e. Ψ crosses the horizontal axis only once. Notice first that the regularity of $F(\theta)$ implies:

$$\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}} < 0 \tag{65}$$

Moreover, when $\Psi(p^*) = 0$ and $p^* \in (c, \overline{\theta})$, by rearranging the terms of (28), we get:

$$\left[\left(\frac{1-F(p^*)}{f(p^*)}\right) - (p^*-c)\right] = \frac{\frac{1-F(p^*)}{f(p^*)}(p^*-c)\left[\frac{iN}{\sigma(N)} + (1-F(p^*)\right]}{(\eta-1)\int_p^{\overline{\theta}}\theta - pdF(\theta)} \equiv \tilde{A} \geq 0$$

and

$$(p^*-c)\left[\frac{iN}{\sigma(N)} + (1-F(p^*))\right] = (\eta-1)\int_p^{\overline{\theta}}\theta - pdF(\theta)\tilde{B}$$

where:

$$\tilde{B} \equiv \frac{\left(\frac{1-F(p^*)}{f(p^*)}\right) - (p^*-c)}{\left(\frac{1-F(p^*)}{f(p^*)}\right)} < 1$$

Condition (64) can be then rewritten as:

$$
\begin{aligned}
& \left.\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}}\right|_{\hat{p}=p^*} \times \\
\times \quad & \left\{ \underbrace{-(p^*-c)\left[\frac{iN}{\sigma(N)}+(1-F(p^*))\right]+(\eta-1)\int_p^{\overline{\theta}}\theta-pdF(\theta)}_{\equiv\left(1-\tilde{B}\right)(\eta-1)\int_p^{\overline{\theta}}\theta-pdF(\theta)>0} \right\} + \\
+ \quad & \left(\frac{1-F(p^*)}{f(p^*)}\right)\left\{ \underbrace{-\left[\frac{iN}{\sigma(N)}+(1-F(p^*))\right]+f(p^*)(p^*-c)}_{\equiv-\frac{iN}{\sigma(N)}-\tilde{A}f(p^*)<0} \right\} + \\
- \quad & \tilde{A}(1-F(p^*))<0
\end{aligned}
\tag{66}
$$

Since the sign of (66) does not depend on the level of p^*, in the range $(c,\overline{\theta})$, this implies that $\Psi(p^*)$ always crosses the horizontal axis with a negative slope, i.e., given the continuity of Ψ it happens only once for $p^* \in (c,\overline{\theta})$.

Finally, plugging the equilibrium value of p^* into equation (26), we obtain the equilibrium value of Ω, which completes the characterisation of the DM equilibrium.

Proof of Proposition 3 When $i \in [i^N, i^C)$ and $\sigma(n)$ exhibits a constant elasticity $(1/\eta)$: (i) p^* does not depend on the nominal rate, which in turn (ii) negatively influences n^*. Proving (i) is trivial since the term in i cancels out from the first order condition (28), once we plug it in condition (29), holding as an equality. To prove (ii), differentiate totally (29) to get:

$$
\frac{dn^*}{di} = -\frac{-p^*}{-\frac{\sigma(n^*)-\sigma'(n^*)n^*}{n^{*2}}\int_{p^*}^{\overline{\theta}}(\theta-p^*)f(\theta)d\theta} < 0 \tag{67}
$$

where $\frac{\sigma(n^*)-\sigma'(n^*)n^*}{n^{*2}} = \left(\frac{\eta-1}{\eta}\right) > 0$.

To prove uniqueness, we follow the same route of the previous proof: we compute $d\Psi(p^*)/dp^*$, then we show that this is always negative.

From (29), holding as an equality, we obtain:

$$
\frac{in}{\sigma(n)} = \frac{\int_{p^*}^{\overline{\theta}}(\theta-p^*)dF(\theta)}{p^*}
$$

Plugging this into (28), evaluated at $\hat{p}=p^*$, and differentiating with respect to $\hat{p}$

we get:

$$
\begin{aligned}
- & \left[\left.\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}}\right|_{\hat{p}=p^*} (p^*-c) + \left(\frac{1-F(p^*)}{f(p^*)}\right)\right] \times \\
\times & \left[\left(\frac{\int_{p^*}^{\overline{\theta}} (\theta-p^*) dF(\theta)}{p^*} \right) + (1-F(p^*)) \right] + \\
+ & \left[-\left(\frac{1-F(p^*)}{f(p^*)}\right)(p^*-c)\right] \left[\tilde{C} - f(p^*)\right] + \\
+ & \left[\left.\frac{d\left((1-F(\hat{p})/f(\hat{p})\right)}{d\hat{p}}\right|_{\hat{p}=p^*} - 1\right] (\eta-1) \int_{p}^{\overline{\theta}} \theta - p dF(\theta) + \\
- & \left[\left(\frac{1-F(p^*)}{f(p^*)}\right) - (p^*-c)\right] (1-F(p^*))
\end{aligned} \tag{68}
$$

where:

$$
\tilde{C} \equiv \frac{-(1-F(p^*))\,(p^*) - \int_{p^*}^{\overline{\theta}} (\theta-p^*) dF(\theta)}{(p^*)^2} < 0 \quad \forall\; p^* \in (c, \overline{\theta})
$$

This implies that, in equation (68), $\left[-\left(\frac{1-F(p^*)}{f(p^*)}\right)(p^*-c)\right]$ is multiplied by a term that is always negative, i.e. $\left[\tilde{C} - f(p^*)\right]$. Hence, as we did in the previous proof, we can rearrange the terms of equation (68) and easily show that the slope of $\Psi(p^*)$ is negative, independently of the level p^*, i.e. function Ψ never crosses the horizontal axes from below. This proves that p^* is unique.

Proof of Proposition 5 Combine (42) and (43) and define:

$$
\begin{aligned}
\Phi(\hat{p}) \;\equiv\; & -\left(\frac{1-F(\hat{p})}{f(\hat{p})}\right)(\hat{p}-c) \left[\frac{i\int_{\hat{p}}^{\overline{\theta}} (\theta-\hat{p}) dF(\theta)}{i\hat{p}+k} + (1-F(\hat{p})) \right] + \\
+ & \left[\left(\frac{1-F(\hat{p})}{f(\hat{p})}\right) - (\hat{p}-c)\right] (\eta-1) \int_{\hat{p}}^{\overline{\theta}} (\theta-\hat{p}) dF(\theta) = 0
\end{aligned} \tag{69}
$$

Notice that $\Phi(c) \equiv \left(\frac{1-F(c)}{f(c)}\right)(\eta-1)\int_{c}^{\overline{\theta}}(\theta-c)dF(\theta) > 0$, $\Phi(\tilde{p}) < 0$, where $\tilde{p}$ was defined in the Proof of Proposition 1. Since $\Phi(\hat{p})$ is continuous, by virtue of the intermediate value theorem, we can conclude that there exists a $\overline{p} \in (c, \overline{\theta})$ that solves $\Phi(\hat{p}) = 0$. To prove uniqueness, we can apply the same argument employed in the proof of Proposition 3 by replacing p^* with $\overline{p}$ and $\tilde{A}$ and $\tilde{C}$ with:

$$
\overline{A} = \frac{\frac{1-F(\overline{p})}{f(\overline{p})}(\overline{p}-c)\left[\frac{in}{\sigma(n)} + (1-F(\overline{p})\right]}{(\eta-1)\int_{p}^{\overline{\theta}}(\theta-\overline{p})dF(\theta)} \geq 0
$$

and

$$\overline{C} \equiv \frac{-i(1-F(\overline{p}))(i\overline{p}+k) - i\int_{\overline{p}}^{\overline{\theta}}(\theta-\overline{p})dF(\theta)}{(i\overline{p}+k)^2} < 0$$

$\forall \overline{p} \in (c, \overline{\theta})$. By applying the implicit function theorem to (69), we finally get that, at the equilibrium:

$$\frac{d\overline{p}}{di} = -\frac{-\left(\frac{1-F(\overline{p})}{f(\overline{p})}\right)(\overline{p}-c)\frac{k\int_{\overline{p}}^{\overline{\theta}}(\theta-\overline{p})dF(\theta)}{(i\overline{p}+k)^2}}{\Phi'(\overline{p})} < 0 \tag{70}$$

Proof of Lemma 4 The proof if structured in two steps.

Step 1. Let consider a seller j deviating and charging p_j. We denote with $n(p_j)$ the buyers/sellers ratio in the submarket j. The profit function of the deviating seller is given by:

$$\Pi_j = \sigma(n(p_j))(p_j - c)(1 - F(p_j)) \tag{71}$$

Nash Equilibrium requires:

$$\sigma'(n(\overline{p}))\left.\frac{dn(p_j)}{dp_j}\right|_{p_j=\overline{p}}(\overline{p}-c)(1-F(\overline{p})) - \sigma(n(\overline{p}))\left[(\overline{p}-c)f(\overline{p}) - (1-F(\overline{p}))\right] = 0 \tag{72}$$

Assuming $f(\overline{p}) > 0$, since $(\overline{p} > c)$, we must have:

$$\sigma'(n(\overline{p}))\left.\frac{dn(p_j)}{dp_j}\right|_{p_j=\overline{p}}(\overline{p}-c)(1-F(\overline{p})) + \sigma(n(\overline{p}))(1-F(\overline{p})) > 0$$

which implies:

$$-\frac{dn(\overline{p})/d\overline{p}}{\overline{p}} < \frac{\sigma(n(\overline{p}))}{\sigma'(n(\overline{p}))(\overline{p}-c)} \tag{73}$$

Step 2. At the equilibrium, buyers must be indifferent between submarkets. At $i = 0$, defining $G(p_j) = \int_{p_j}^{\overline{\theta}}(\theta - p_j)dF(\theta)$, this implies:

$$\frac{\sigma(n(p_j))}{n(p_j)}G(p_j) = \frac{\sigma(\overline{n})}{\overline{n}}G(\overline{p}) \tag{74}$$

Differentiating totally (74) with respect to p_j, we get:

$$\frac{\sigma(n(p_j)) - \sigma'(n(p_j))n(p_j))}{[n(p_j)]^2}\frac{dn(p_j))}{dp_j}G(p_j) - \frac{\sigma(n(p_j))}{n(p_j)}(1-F(p_j)) = 0 \tag{75}$$

which must hold $\forall\ p_j$ and so also for $p_j = \overline{p}(0)$. Equation (75) can be then rewritten as:

$$\left[1 - \frac{\overline{n}\sigma'(\overline{n})}{\sigma(\overline{n})}\right]\left[-\frac{dn(\overline{p}(0))/d\overline{p}(0)}{\overline{p}(0)}G(\overline{p}(0))\right] = (1 - F(\overline{p}(0))) \tag{76}$$

Using condition (73), equation (76) becomes:

$$\left[1-\frac{\overline{n}\sigma'(\overline{n})}{\sigma(\overline{n})}\right]\left[\frac{\sigma(n(\overline{p}(0)))}{\sigma'(\overline{n})\overline{n}(\overline{p}(0)-c)}G(\overline{p})\right] > (1-F(\overline{p}(0)))$$

which can be rewritten as:

$$\left[1-\frac{\overline{n}\sigma'(\overline{n})}{\sigma(\overline{n})}\right]\left[\frac{\sigma(n(\overline{p}(0)))}{\sigma'(\overline{n})\overline{n}}G(\overline{p}(0))\right] > (\overline{p}-c)(1-F(\overline{p}(0))) \tag{77}$$

where the first term in square brackets is $[1-1/\eta]$, since we assume that $\sigma(n)$ exhibits constant return to scale, with elasticity $1/\eta$. Rearranging terms, and recalling that $G(\overline{p})+(\overline{p}(0)-c)(1-F(\overline{p}(0)))=\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-c)dF(\theta)$, we finally get:

$$\frac{\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-\overline{p}(0))dF(\theta)}{\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-c)dF(\theta)} \equiv \frac{\overline{n}/\sigma(\overline{n})k}{\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-c)dF(\theta)} > \frac{1}{\eta} \tag{78}$$

Proof of Proposition 6 Given equation (43), aggregate welfare can be rewritten as:

$$\mathcal{W}=\frac{1}{1-\beta}\left[\Pi(i)+ip(i)\overline{n}\right] \tag{79}$$

where

$$\Pi(i)\equiv\sigma(\overline{n})\int_{\overline{p}(i)}^{\overline{\theta}}\left(\overline{p}(i)-c\right)dF(\theta)$$

and $\overline{n}$ is implicitly defined by (43). The first order condition of the monetary authority is:

$$\frac{d\mathcal{W}}{di}=\frac{1}{1-\beta}\left[\frac{\partial\Pi(i)}{\partial\overline{p}}\frac{\partial\overline{p}}{\partial i}+\frac{\partial\Pi(i)}{\partial i}+\frac{d}{di}\left(i\overline{p}(i)\overline{n}\right)\right]=0 \tag{80}$$

Notice that, from the seller's problem optimality condition (42)-(43), we have $\partial\Pi(i)/\partial\overline{p}=0$. Condition (80) reduces then to:

$$\frac{d\mathcal{W}}{di}=\sigma'(\overline{n})\frac{\partial\overline{n}}{\partial i}\left(\overline{p}(i)-c\right)\left(1-F(\overline{p})+\overline{p}\overline{n}+i\overline{n}\frac{d\overline{p}}{di}+\frac{d\overline{n}}{di}i\overline{p}=0\right. \tag{81}$$

Evaluating (81) at $i=0$, we get:

$$\left.\frac{d\mathcal{W}}{di}\right|_{i=0}=\sigma'(\overline{n})\left.\frac{\partial\overline{n}}{\partial i}\right|_{i=0}\left(\overline{p}(0)-c\right)\left(1-F(\overline{p}(0))+\overline{p}(0)\right. \tag{82}$$

where:

$$\left.\frac{\partial\overline{n}}{\partial i}\right|_{i=0}=-\frac{\overline{p}(0)}{\left(\frac{\sigma(\overline{n})-\sigma'(\overline{n})\overline{n}}{\overline{n}^2}\right)\int_{\overline{p}(0)}^{\overline{\theta}}\left(\theta-\overline{p}(0)\right)dF(\theta)} \tag{83}$$

Plugging (83) into (82) we obtain:

$$\begin{aligned}
\left.\frac{d\mathcal{W}}{di}\right|_{i=0} &= \overline{p}(0)\left[1-\frac{1}{\eta-1}\frac{(\overline{p}(0)-c)\,(1-F(\overline{p}(0))}{\sigma(\overline{n})/\overline{n}\int_{\overline{p}(0)}^{\overline{\theta}}(\theta-\overline{p}(0))dF(\theta)}\right] \\
&= \overline{p}(0)\left[1-\frac{1}{\eta-1}\frac{(\overline{p}(0)-c)\,(1-F(\overline{p}(0))}{k}\right] \qquad (84)
\end{aligned}$$

with $\eta-1>0$. Since $\overline{p}(0)>0$, we see that $d\mathcal{W}/di$ is positive at $i=0$ if:

$$\frac{(\eta-1)k-(\overline{p}(0)-c)\,(1-F(\overline{p}(0))}{(\eta-1)k}>0$$

Lemma 4 ensures that this inequality holds.